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Quantum Hall electrodynamics: classical and relativistic aspects *

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Abstract

Physics of any system is governed by a set of conservation laws which depends on the system dimensionality. A typical case are electrons in external magnetic filed. The central players here are the adiabatic invariance of the orbital magnetic moment of an electron in a magnetic field in classical physics and its quantum counterpart – the Landau levels.

These general principles have recently attracted growing interest due to the puzzling discovery by R. Mani and A. Zudov's groups of

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nondissipative structures in two-dimensional electron systems under the classically strong magnetic fields $\omega_c\tau \gg 1$.

It may well be that this quantum-classical controversy is rather in terminnology, and the origin of physical mysteries of a magnetized two-dimensional electron gas is of more general, probably, topological origin, and follows from the internal symmetries of a two-dimensional electron gas in a magnetic field.

While the experimental observation of the wave frequency dependent "windows" of vanishing magnetoresistivity are by now widely discussed in the literature, the aim of this review is to collect some already known information on the hidden symmetries of these systems, like relaivistic and scaling invariance, and to apply to them the Chern-Simons electrodynamics.

The relativisticaly invariant electrodynamics of quantum Hall media is discused and some new solutions are presented, generic to the two-dimensional electron system under magnetic field in nondissipative regime.

We review here also some previous works related to the classical-quantum finite frequency responce of the two-dimensional electron systems under a wide range of magnetic fields. The phenomenon of zero resistivity in these systems is instructive also as a rich enough *toy model* to study the basic laws of statistical physics, leading to fundamental understanding of Irreversibility. This is discussed in details in terms of the Basic Irreversible Cycle for Electrons in Magnetic Fields (BIC EMF). The helicon mode is chosen as a good example of an electromagnetic wave propagating due to the self-consistency of the electromagnetically induced Hall currents in the sample. The notion of the quantum-Hall-helicon mode is introduced.

Design and feasibility of relevant experiments are discussed.

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1 Introduction

Two decades after discovery of the quantum Hall effect (QHE) [1–3] revealed a rich world of wonderful physics contained in two dimensional electron gas (2DEG) under strong magnetic fields at low temperatures. Large amount of experimental and theoretical work on this subject is done and undoubtely correct models and theories on different aspects of the phenomena have been constructed. Neverthless, these system were and still are surrounded by some mistery.

A good example is the new anomalies in the microwave absorbtion in QHE systems revealed recently in the ingeneous experiments in the groups of R. Mani [4] and A. Zudov [5].

The puzzle is in the fact that the famous QHE zero resistance states, the manifestation of the quantization of the Landau levels, $\hbar\omega_c(t) >> k_B T$, here $\omega_c(t) = eH/m^*c$ is the cyclotron frequency, (e, H, m^*, c are the electron charge, external magnetic field,electron effective mass, and light velocity, respectively), appear now in the apparently classical regime of magntoresistance, i.e. at sample parameters where thermal smearing of the Landau levels is larger than the distance between two adjasent Landau levels – the Landau gap. This corresponds to the classically strong magnetic fields: $\omega_c\tau \gg 1$, here τ is the electron relaxation time.

These experimental findings have stimulated recently a number of different theoretical models [6]. The possibility of *absolute negative conductivity*, which is commonly believed to be among the main mechanisms leading to the vanishing magnetoresistance have been considered for the first time by Ryzhii already in 1969 [7]. More recent theoretical [8–16] and experimental [17–21] studies of the finite frequency responce of QHE systems have been active in 1980-1990-th and some insight have been gathered at that time. Nothing unexpected has been observed and until recent discoveries of the Mani and Zudov groups the subject was practically abandoned.

The aim of this paper is to review some previous work related to the classical-quantum finite frequency responce of the two-dimensional electron systems under a wide range of magnetic fields.

In Sect. 2 the relativisticaly invariant electrodynamics of QHE medium is discused and some new solutions are presented. It may well be that this quantum-classical controversy is rather in terminnology, and the origin of the 2DEG+H physical mysteries is of more general, probably, topological origin and follows from the internal symmetries, notably the *Chern-Simons* one, generic to the two-dimensional electron system under magnetic field in nondissipative regime.

The phenomenon of zero resistivity in these systems is instructive also as a rich enough *toy model* to study the basic laws of statistical physics, leading to fundamental understanding of Irreversibility. This is discussed in detailes in Sect. 3 in terms of the Basic Irreversible Cycle for Electrons in Magnetic Fields (BIC EMF), and design and feasibility of relevant experiments are presented.

1.1 Relativistic invariance of quantum Hall effect

Relativistic and scaling invariance of the QHE can be obtained from the following simple phenomenological consideration [16].

Under very general conditions, in the theory of continuous media the induced charges and currents are linearly connected with electric and magnetic fields:

$$J_\mu = \Xi_{\mu\nu\rho} F^{\nu\rho} \tag{1}$$

where j_μ is the current, $F_{\nu\rho}$ is the electromagnetic field strength tensor, and $\Xi_{\mu\nu\rho}$ are the coefficients, defining the physical properties of a conducting medium. The coefficients express the linear response of the medium (conductor, insulator, ferromagnet, etc.) to applied fields. In all usual condensed matter cases medium destroys relativistic invariance of Maxwell equations. In vacuum, which is a relativistically invariant medium, $\Xi_{\mu\nu\rho}$ is zero. If the coefficients $\Xi_{\mu\nu\rho}$ form an invariant tensor: $\Xi'_{\mu\nu\rho} = \Xi_{\mu\nu\rho}$ where prime corresponds to a moving frame, the medium would preserve the relativistic invariance. In D + 1 space-time dimensions the only constant tensors are the metric tensor $g_{\mu\nu}$ and the totally antisymmetric D + 1 component tensor $\epsilon_{\mu\nu}$ ... . In 3 + 1 dimensions it is impossible to construct a nonvanishing constant three-component tensor, out of $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. Therefore the only relativistically invariant medium in 3+1 dimensions is vacuum, where $\Xi_{\mu\nu\rho\sigma} = 0$.

However, in 2 + 1 dimensions such a tensor may exist: $\sigma_{\mu\nu\rho} \propto \epsilon_{\mu\nu\rho}$. Let us show that this situation is realized under the QHE conditions.

Indeed, in the QHE medium (in the plateau regime) the material equations are:

$$j_1 = \sigma_{xy} E_y \equiv \nu\alpha E_2; \qquad j_2 = -\sigma_{xy} E_x \equiv -\nu\alpha E_1 \tag{2}$$

where ν is the filling factor and α is the fine structure constant. These two equations for the Hall currents could be cast in the form:

$$j_i = \nu \epsilon_{ij0} E^j \tag{3}$$

where $i, j = 1, 2$. The zero component of current

$$j_0 = \nu\alpha B_3 \tag{4}$$

is, in fact, the Faraday law for a nondissipative medium: $\int j_0 \propto Q \equiv \nu\alpha \int B_z$.

Eqs. (3),(4) at $\nu = 1$, could be cast in the form:

$$j^i = 2c\alpha e^{ijk} F_{jk} \tag{5}$$

Eq. (5) follows also from the expression action:

$$S_A \propto \int d^3x \left[-\frac{1}{c} j^i A_i + \alpha e^{ijk} A_i F_{jk} \right].$$

Indeed, the variation of this action:

$$\delta S_A \propto \left[-\frac{1}{c} j^i A_i + 2\alpha e^{ijk} A_i F_{jk} \right] \delta A_i = 0 \tag{6}$$

yields Eq. (5).

The mentioned above Lorentz invariance of QHE is connected with the Chern-Simons term, which is a new, compared to 3D electrodynamics, term in Lagrangian.

Inserting Eqs. (3),(4) into the effective Maxwell equations, we obtain the macroscopic equations in medium in Fourier space [9, 16]:

$$k_\mu F_{\mu\nu} = \nu\alpha\epsilon_{\mu\nu\rho}\sqrt{k^2}F_{\nu\rho}; \qquad \epsilon_{\mu\nu\rho}\partial_\mu F_\nu \rho = 0 \tag{7}$$

or

$$\partial^\mu F_{\mu\nu} = \nu\alpha\epsilon_{\mu\nu\rho}\sqrt{\partial^2}F^{\nu\rho}. \tag{8}$$

The Lagrangian generating these equations is:

$$\mathcal{L} = \frac{1}{4} f_{\alpha\beta} \frac{\theta(\partial_\alpha \partial^\alpha)}{\sqrt{\partial_\alpha \partial^\alpha}} f^{\alpha\beta} + \nu \epsilon_{\alpha\beta\gamma} a^\alpha f^{\beta\gamma} \tag{9}$$

where the first term is the vacuum Lagrangian, and the second is the Chern-Simons term.

It can be easily shown explicitly by performing Lorentz transformation for the currents and fields, that the nondiagonal components of the conductivity tensor preserves the Lorentz invariance while the diagonal components destroy it. This will be elaborated in details in what follows.

1.2 Composite Fermions-Chern-Simons gauge transformed particles

It is known that the integer quantum Hall effect (IQHE) can be qualitatively understood on the basis of single electron wave functions in the electrostatic potential of the long range fluctutions in the sample, the fractional quantum Hall effect (FQHE) is essentially a many-body phenomenon, and was extensively studied, mainly in the framework of the "Laughlin wave function" formalism [22, 23].

A very popular model of *Composite Fermions* was sugested by Jain [24, 25], who used a very transparent picture of magnetic fluxes attached to an electron, in the spirit of the Faraday magnetic flux lines piercing the conducting media.

The vector potential associated with this flux tube (solenoid) may be written as:

$$\vec{a}(\vec{r}) = \frac{\Phi}{2\pi} \frac{\vec{z} \times \vec{r}}{r^2} \tag{10}$$

where $\vec{z}$ is a unit vector along the z - axis. The magnetic field B, associated with such vector potential, is that of a vortex of strength Φ, localized at the origin:

$$\vec{B} = \mathrm{curl}\vec{a} = \Phi\delta(\vec{r}) \tag{11}$$

where the δ-function arises from the singularity, at $\vec{r} = 0$ of the vector potential, Eq. (10). The magnetic flux connected to such field is:

$$\int \vec{B} d\vec{s} = \Phi. \tag{12}$$

The Schrödinger equation for a particle in such a field, is:

$$\hat{H}(\Phi)\psi(r) = \frac{1}{2m}\left(\vec{\mathcal{P}} - \frac{e}{c}\vec{a}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}) \tag{13}$$

where the canonical momentum operator $\vec{\mathcal{P}}$ is:

$$\vec{\mathcal{P}} = -i\hbar\frac{\partial}{\partial\vec{r}}. \tag{14}$$

The wave function $\psi(\vec{r})$ should be periodic under rotations on the angle $\theta = 2\pi$ around z-axis.

For the flux-tube-particle composite, the spectrum of the angular momentum is:

$$\ell_z = \hbar\left(m - \frac{\Phi}{\Phi_0}\right) \tag{15}$$

Following Wilczek, [26], the angular momentum of this composite particle is equal to

$$s = \frac{\ell_z(m=0)}{\hbar} = -\frac{\Phi}{\Phi_0} \tag{16}$$

In general, s is neither integer nor half integer, and can take any value. If we interchange flux-tube-particle composites, we will have an additional phase factor. Since the interchange of two such composites can give *any* phase, Wilczek called them *anyons*.

For a system of N flux-tube-particle composites, the charged particle feels the vector potential of flux tubes, "glued" to all the other particles. The corresponding Hamiltonian can be written as follows:

$$\hat{H} = \frac{1}{2m}\sum_{i=1}^{N}\left(\vec{\mathcal{P}} - \frac{e}{c}\vec{a}_i(\vec{r}_i, \ldots \vec{r}_N)\right)^2 \tag{17}$$

with the vector potential:

$$\vec{a}_i = \frac{\Phi}{2\pi} \sum_{j \neq i} \frac{\vec{z} \times (\vec{r}_i - \vec{r}_j)}{(\vec{r}_i - \vec{r}_j)^2} = \frac{\Phi}{2\pi} \sum_{j \neq i} \nabla \theta_{ij} \tag{18}$$

where θ_{ij} is the angle between the vector $\vec{r}_i - \vec{r}_j$ and the x-axis. This effective vector potential is nonlocal, since it depends on the position of all particles and, in particular, vanishes at $N = 1$. The Hamiltonian, Eq. (17), describes the interaction of charged particles with the Chern-Simons gauge field [27], and this vector-potential corresponds to the following effective magnetic field:

$$\vec{B}'_i = \vec{B} - \Phi \sum_{j \neq i} \delta(\vec{r}_i - \vec{r}_j) \tag{19}$$

which can be presented as

$$B_{eff} = B - \frac{\Phi}{\Phi_0} \frac{2\pi\hbar c}{e} n_e = B \left(1 - \frac{\Phi}{\Phi_0} 2\pi \ell_B^2 n_e \right) = B \left(1 - \frac{\Phi}{\Phi_0} \nu \right). \tag{20}$$

As it follows from Eq.(20), the effective magnetic field B_{eff} is equal to zero for the magnetic flux $\Phi = \frac{\Phi_0}{\nu}$. The case of the filling factor $\nu = \frac{1}{2n}$ is of special interest, because for $\Phi = 2n\Phi_0$ the statistics of particles is not changed. A very interesting situation corresponds to the filling factor $\nu = \frac{1}{2}$, when the dimensionless magnetic flux

$$\tilde{\Phi} = \frac{\Phi}{\Phi_0} \tag{21}$$

is equal to 2. The completely filled Landau level with $m = 1$ and $B_{eff} = 0$ corresponds to $\tilde{\Phi} = 1$.

1.3 Magneto-density oscillations

Let us give some classical insight into the flux-electron coupling, typical for the ideal (nondissipative) plasma physics. The magnetic flux is frozen, i.e. the charges are "glued" to the magnetic field lines, if the time variation of the magnetic field, or charge density, is slow enough. In 2DEG+H, axially symmetric fluctuation of the magnetic flux $\delta\Phi(t, \mathbf{r})$ will drive a charge density fluctuation $\delta n(t, \mathbf{r})$ which, in turn, will create a radial electric field $\mathbf{E}(t, \mathbf{r})$. Finally, a magnetic moment $\delta\mathbf{M}(t, \mathbf{r})$ will be added to the initial magnetic field, due to the azimutal Hall currents: $j_\phi(t, \mathbf{r}) \propto E_r H_z$. This may result in an oscillatory process, i.e. in a periodic transformation of the magnetic energy into the kinetic energy of Hall currents and vice versa, if the energy, stored in such fluctuation, will dissipate (due to the Joule heat produced by the radial currents $W \propto \int j_r E_r$) slow with respect to the oscillation period. Since in QHE the diagonal conductivity is, practically, zero, one may expect unusual for a 3D electron system in solids electrodynamics.

Consider the evolution of a density fluctuation δn(t,**r**) in an initially homogeneous 2DEG with the areal density n_o in a homogeneous external magnetic field along the z-direction $H = H_z$. We start with the continuity equation:

$$-\frac{ed\delta n}{dt} + \mathrm{div}\mathbf{j} = 0 \tag{22}$$

where the current is defined as:

$$j_i = \sigma_{ij} E_j. \tag{23}$$

Assuming nondissipative regime σ_{xx}= 0, Eq. (23) reads:

$$-\frac{ed\delta n}{dt} = \mathrm{div}(\sigma_{xy}\left[\delta\mathbf{E}\times\mathbf{h}\right]) = -\sigma_{xy}\mathbf{h}\mathrm{curl}\delta\mathbf{E} + \left[\delta\mathbf{E}\times\mathbf{h}\right]\mathrm{grad}\sigma_{xy}.$$

Using the Faraday law:

$$c \;\; \mathrm{curl}\mathbf{E} = -\frac{d\mathbf{B}}{dt}, \tag{24}$$

we arrive at:

$$-\frac{ed\delta n}{dt} = -\frac{\sigma_{xy}}{c}\frac{d\delta B}{dt} + \left[\delta\mathbf{E}\times\mathbf{h}\right]\cdot\mathrm{grad}\sigma_{xy}. \tag{25}$$

Neglecting the nonlinear term (the second term in the right side), we arrive at the proportionality between the magnetic flux and the charge density:

$$\delta B = -\frac{ec}{\sigma_{xy}}\delta n = -i\frac{e^2}{h}\delta n. \tag{26}$$

The proportionality of the electron density and the magnetic flux in a fluctuation described above, is depicted in Fig. 1.

It follows from Eq. (25) that vanishing of the diagonal components of the conductivity tensor $\sigma_{xx} = 0$ ensures that the Hall conductivity σ_{oxy}, the only material parameter entering the linearized equations, is a constant of motion. Moreover, the QHE fixes this constant at a universal value ie^2/h, (i is an integer or a simple fraction p/q), thus giving a universal relationship between the charge density and the magnetic flux fluctuations.

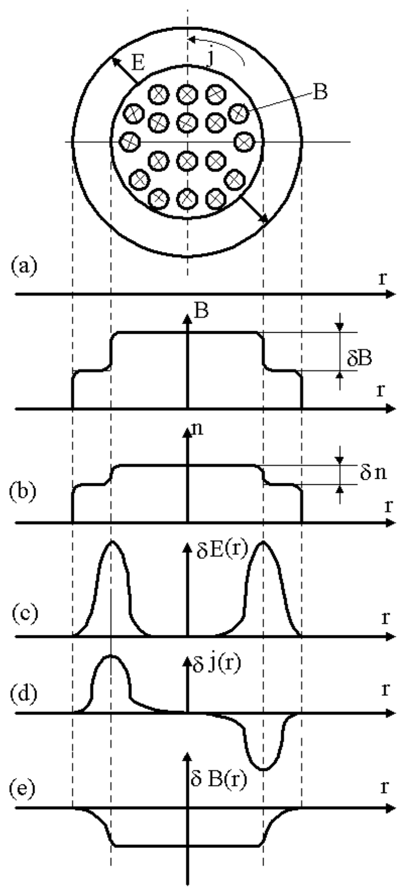

Figure 1: An axially symmetric fluctuation of the magnetic flux $\delta\Phi(t, \mathbf{r})$ will drive a charge density fluctuation $\delta n(t, \mathbf{r})$ which, in turn, will create a radial electric field $\mathbf{E}(t, \mathbf{r})$. Finally, a magnetic moment $\delta\mathbf{M}(t, \mathbf{r})$ will be added to the initial magnetic field, due to the azimutal Hall currents: $j_\phi(t, \mathbf{r}) \propto E_r H_z$.

2 Quantum Hall electrodynamics

In this section we develop the electrodynamics of a purely two-dimensional electron system under strong magnetic field. At sufficiently low temperatures these systems are in a nondissipative QHE state at integer (IQHE) or certain fractional (FQHE) filling factors. We follow closely the presentation in [9, 16, 28] and present some of the consequences of the inherent symmetries of QHE medium on the in-plane electromagnetic wave propagation.

We describe here the effective electrodynamics of currents and charges in the plane in two-dimensional electron systems. In QHE systems all the possible solutions of Maxwell equations on the $x_3 = 0$ plane are found for components E_1 and E_2 of electric field and component B_3 of magnetic field.

Introducing the field potentials $\overrightarrow{A}$ and Φ via

$$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}, \tag{27}$$

$$\overrightarrow{E} = -\frac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\Phi, \tag{28}$$

and using the Lorenz gauge

$$\partial^{\mu} A_{\mu} = \frac{\partial \Phi}{\partial t} + \overrightarrow{\nabla}\overrightarrow{A} = 0 \tag{29}$$

where A_{ν} are the components of electromagnetic potentials $A_{\mu} = (\Phi, \overrightarrow{A})$; $X^{\mu} = (X_0 = t, \overrightarrow{X})$,we arrive at Maxwell equations in the form of four independent D'Alambert equations

$$\Box A_{\mu} = -4\pi J_{\mu}, \tag{30}$$

where J_{μ} are the components of currents, $J^{\mu} = (\varrho, \overrightarrow{J})$, ($J^0 = \varrho$ is the charge density), $(\mu = 0, 1, 2, 3)$ and $\Box \equiv \Delta - \frac{\partial^2}{\partial t^2}$.

Any solution of these equations can be represented in the form

$$A_{\mu}(X) = \int G(X-Y) 4\pi J_{\mu}(Y) dY + A^0_{\mu}(X), \tag{31}$$

where $dY \equiv dY_0 dY_1 dY_2 dY_3$. Here $A^0_{\nu}(X)$ is a solution of the free D'Alamber equation and $G(X)$ is the Green function defined by

$$\Box G(X) = -\delta(X). \tag{32}$$

Of course, the free potentials $A^0_{\mu}(X)$ have to satisfy the Lorenz gauge too.

The choice of Green function is defined by boundary conditions, which are dependent on the physical aspects of the problem. We choose here the symmetric Green function [29] $G(X) = D^{sym}(X)$:

$$D^{sym}(X) = \int dK D^{sym}(K) \exp(iKX) = \frac{1}{(2\pi)^4} \mathcal{P} \int dK \frac{\exp(iKX)}{K^2},$$

here $KX = K_0X_0 - K_1X_1 - K_2X_2 - K_3X_3; K^2 \equiv K_1^2 + K_2^2 + K_3^2 - K_0^2$ and $dK \equiv dK_0K_1dK_2dK_3$.

2.1 Charges and currents confined to a plane.

In the case of 2DEG layer ($X_3 \equiv 0$) we assume that

$$J_\alpha(X) = \delta(X_3) j_\alpha(x, t), \tag{33}$$

where ($\alpha = 0, 1, 2$), $x \equiv x_1, x_2$ and the component j_3 of the current is equal to zero

$$J_3 \equiv 0.$$

The potentials $A_\mu(x, t)$ on the plane ($X_3 = 0$) are

$$a_\mu(x, t) \equiv A_\mu(x, x_3 = 0, t) = \int dy D^{sym}(x - y, 0, \tau) 4\pi j_\mu(y) + A_\mu^0(x, x_3 = 0) \tag{34}$$

where $\tau = x_0 - y_0$. Since $j_3 \equiv 0$, we have $A_3(X) \equiv A_3^0(X)$ and the Lorenz gauge condition on the plane has the form

$$\sum_{\alpha=0,1,2} \partial^\alpha a_\alpha + \eta_3 = 0 \tag{35}$$

where $\eta_3(x, t) = \partial^3 A_3^0(x, x_3 = 0, t)$.

For free potentials the gauge condition has the same form

$$\sum_{\alpha=0,1,2} \partial^\alpha a_\alpha^0 + \eta_3 = 0. \tag{36}$$

In the Fourier space

$$(k^2 - \omega^2) a_\alpha(\omega, k) = 4\pi j_\alpha^{eff}(\omega, k) + (k^2 - \omega^2) a_\alpha^0(\omega, k), \tag{37}$$

$$j_\alpha^{eff}(\omega,k) = j_\alpha(\omega,k) f(\omega, k)/2. \tag{38}$$

Here $f(\omega, k) = \theta(k^2 - \omega^2)\sqrt{k^2 - \omega^2}$.

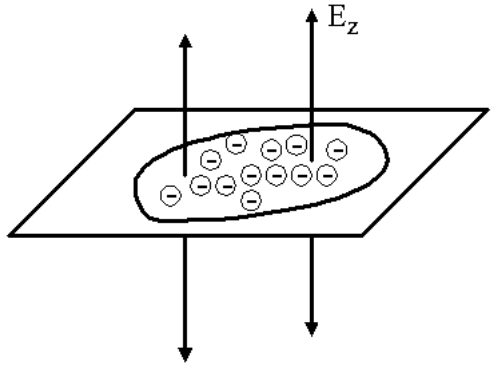

Figure 2: Electrostatics of a 2D charge distribution.

It means that

$$\partial_\beta \partial^\beta a_\alpha(x) = 4\pi j_\alpha^{eff}(x) + \partial_\beta \partial^\beta a_\alpha^0(x). \tag{39}$$

It is important to note that in k-space the set of points such that $(k^2-\omega^2)$ $a_\alpha^0(\omega, k) \neq 0$ satisfies the condition $k^2 < \omega^2$, i.e. $(k^2-\omega^2)a_\alpha^0(\omega, k) \sim \theta(\omega^2 - k^2)$. Indeed, $A_\mu^0(\omega, k, k_3) \sim \delta(\omega^2 - k^2 - k_3^2)$ if $k^2 > \omega^2$ then $A_\mu^0(\omega, k, k_3) = 0$.

The functions $a_\alpha^0(\omega, k)$ can be obtained by integration $A_\mu^0(\omega, k, k_3)$ over k_3. So when $k^2 > \omega^2$, one gets $a_\alpha^0(\omega, k) = 0$.

Effective Maxwell equations in x-space for the field components E_1, E_2 and B_3 on the plane $x_3 \equiv 0$ can be obtained from (39) and (35), (36)

$$\partial_1 \widetilde{E}_1 + \partial_2 \widetilde{E}_2 = 4\pi j_o^{eff}, \tag{40}$$

$$-\partial_t \widetilde{E}_1 + \partial_2 \widetilde{B}_3 = 4\pi j_1^{eff}; \tag{41}$$

$$-\partial_t \widetilde{E}_2 - \partial_1 \widetilde{B}_3 = 4\pi j_2^{eff}; \tag{42}$$

$$\partial_1 E_2 - \partial_2 E_1 = -\partial_t B_3 \tag{43}$$

where $\widetilde{B} \equiv B - B_0$ and $\widetilde{E} \equiv E - E_0$. Here E_α^0, B_3^0 are the components of some free field solution.

These equations contain only components E_1, E_2 and B_3 while the components E_3, B_1, B_2 are removed from them completely.

In the Fourier space for Eqs. (40)–(43): one gets

$$-ik_1\widetilde{E}_1 - ik_2\widetilde{E}_2 = 2\pi f(k) j_0(k_1,k_2) \tag{44}$$

$$-i\omega\widetilde{E}_1 - ik_2\widetilde{B}_3 = 2\pi f(k) j_1(k_1,k_2); \tag{45}$$

$$-i\omega\widetilde{E}_2 + ik_1\widetilde{B}_3 = 2\pi f(k) j_2(k_1,k_2); \tag{46}$$

$$-ik_2 E_1 + ik_1 E_2 - i\omega B_3 = 0. \tag{47}$$

2.2 Electrostatics

The case $\omega = 0$ corresponds to a stationary nondissipative solution under QHE conditions. Equation (40) defines connection between electrical field and electrical charge density in the limit $\omega \to 0$. Eq. (43)

$$\partial_2 E_1(x_1,x_2) - \partial_1 E_2(x_1,x_2) = 0$$

gives the potential condition for electrical field. It means that the electrical field is defined as the gradient of the electrical potential $\Phi(x_1,x_2)$:

$$E_\alpha(x_1,x_2) = -\frac{\partial\Phi(x_1,x_2)}{\partial x_\alpha}$$

or in the Fourier space

$$E_\alpha(k_1,k_2) = ik_\alpha\Phi(k_1,k_2). \tag{48}$$

The magnetic field in the Fourier space is

$$ik_\alpha B_3(k_1,k_2) = -2\pi k\sigma E_\alpha(k_1,k_2). \tag{49}$$

The charge density in the static case is connected with the magnetic field B_3 by relation (40)

$$j_0^{(s)}(x_1,x_2) = -\frac{B_3(x_1,x_2)}{(2\pi)^2\sigma}. \tag{50}$$

Let us introduce the "vector" definition for field components

$$\mid\Psi\rangle = \begin{pmatrix} E_1 \\ E_2 \\ B_3 \end{pmatrix}. \tag{51}$$

The electrostatic solution ($\omega = 0$) has the form

$$| \Psi^{(s)}(k_1, k_2)\rangle = - \begin{pmatrix} ik_1 \\ ik_2 \\ \beta k \end{pmatrix} \Phi(k_1, k_2), \tag{52}$$

where $\Phi(k_1, k_2)$ is an arbitrary potential.

It means that for any potential function $\Phi(x_1, x_2)$ on the plane ($x_3 = 0$), the distribution of the components is

$$\begin{aligned} E_i(x_1, x_2) &= -\frac{\partial \Phi(x_1, x_2)}{\partial x_i}, &(53)\\ B_3(x_1, x_2) &= \sigma \int\int_{-\infty}^{+\infty} dk_1 dk_2 \exp[i(k_1 x_1 + k_2 x_2)] k \Phi(k_1, k_2). &(54) \end{aligned}$$

$$\Phi(k_1, k_2) = (2\pi)^{-1} \int\int_{-\infty}^{+\infty} dx_1 dx_2 \exp[i(k_1 x_1 + k_2 x_2)] \Phi(x_1, x_2).$$

The distributions of currents and charge connected with the distributions of the field E_i, B by Eq. (50) stable with time.

If $\Phi(x_1, x_2)$ is independent of x_2, then

$$\Phi(x_1, x_2) \equiv \Phi(x_1), \tag{55}$$

$$E_1(x_1, x_2) \equiv E_1(x_1) = -\frac{\partial \Phi(x_1)}{\partial x_1},\ E_2 = 0 \tag{56}$$

and $B_3(x_1, x_2) \equiv B_3(x_1)$ is defined by formula (53)

For currents $J_1(x_1, x_2) \equiv 0$ and

$$J_2(x_1, x_2) \equiv J_2(x_1) = -\sigma E_1(x_1). \tag{57}$$

2.2.1 Rotational symmetry

For investigation of the rotational symmetry solutions it is useful to use the cylindric coordinates (r, φ).

If $\overrightarrow{x} = (x_1, x_2)$ are Cartesian coordinates on the plane, then

$$x_1 = r\cos(\varphi), x_2 = r\sin(\varphi). \tag{58}$$

The field $\overrightarrow{F}(\overrightarrow{x}) = (F_1(\overrightarrow{x}), F_2(\overrightarrow{x}))$ in cylindric coordinates has the components $F_r(r,\varphi), F_\varphi(r,\varphi)$

$$F_r = F_1 \cos(\varphi) + F_2 \sin(\varphi),$$

$$F_\varphi = -F_1 \sin(\varphi) + F_2 \cos(\varphi).$$

If function $\Phi(r)$ depends on r only, then after Fourier transformation $\Phi(k)$ depends on $k = \sqrt{k_1^2 + k_2^2}$ only.

$$\Phi(k_1, k_2) = (2\pi)^{-1} \int\int_{-\infty}^{+\infty} dx_1 dx_2 \exp[i(k_1x_1 + k_2x_2)]\Phi(r). \tag{59}$$

After the definition $k_1x_1 + k_2x_2 = kr\cos\theta$ we get $dx_1dx_2 = rdrd\theta$

$$\begin{aligned} \Phi(k_1, k_2) &= (2\pi)^{-1} \int_0^\infty rdr \int_0^{2\pi} d\theta \exp[i(kr\cos\theta)]\Phi(r) \\ &= \int_0^\infty rdr J_0(kr)\Phi(r) = \Phi(k). \end{aligned} \tag{60}$$

So if potential depends on r, only then the field B_3 also depends on r only

$$B_3(k) = -k\beta\Phi(k).$$

The electric field.

$$E_1(r,\varphi) = -\frac{\partial\Phi(r)}{\partial x_1} = -\frac{\partial r}{\partial x_1}\frac{\partial\Phi(r)}{\partial r} = -\frac{x_1}{r}\frac{\partial\Phi(r)}{\partial r} = -\cos(\varphi)\frac{\partial\Phi(r)}{\partial r},$$

$$E_2(r,\varphi) = -\frac{\partial\Phi(r)}{\partial x_2} = -\frac{\partial r}{\partial x_2}\frac{\partial\Phi(r)}{\partial r} = -\frac{x_2}{r}\frac{\partial\Phi(r)}{\partial r} = -\sin(\varphi)\frac{\partial\Phi(r)}{\partial r},$$

and in this case

$$E_r = -\frac{\partial\Phi(r)}{\partial r},$$

$$E_\varphi = -E_1 \sin(\varphi) + E_2 \cos(\varphi) = 0.$$

So

$$\mid \Psi^{(s)}(r)\rangle = \begin{pmatrix} E_r \\ E_\varphi \\ B_3 \end{pmatrix} = -\begin{pmatrix} E_r(r) \\ 0 \\ B_3(r) \end{pmatrix}. \tag{61}$$

For the currents on the Plane under QHE conditions we have $j_1(r,\varphi) = \sigma E_2$ $= -\sigma\sin(\varphi)\frac{\partial\Phi(r)}{\partial r}$; $j_2 = \sigma\cos(\varphi)\frac{\partial\Phi(r)}{\partial r}$; $j_r(r) = j_1\cos(\varphi) + j_2\sin(\varphi) = 0$; $j_\varphi(r) = -j_1\sin(\varphi) + j_2\cos(\varphi) = -\sigma E_r(r)$, and $j^s(r) = (j_r, j_\varphi) = (0, j_\varphi(r))$.

Thus the localized stable states with rotational symmetry do exist. The currents circulate around the center of symmetry.

2.2.2 Rotational symmetry superposition

We can construct rotational symmetry superposition of plane waves. The circular currents are absent in this case:

$$\begin{aligned} E_1^\omega(r,\varphi) &= \frac{\partial}{\partial x_1} b(r) = \frac{x_1}{r}\partial_r b(r), \\ E_1^\omega(r,\varphi) &= -\frac{\partial}{\partial x_2} b(r) = -\frac{x_2}{r}\partial_r b(r), \end{aligned}$$

where $b(r)$ is a function of r. By using (58) one obtains

$$E_r^\omega(r,\varphi) = E_1^\omega(r,\varphi)\cos(\varphi) + E_2^\omega(r,\varphi)\sin(\varphi) = 0. \tag{62}$$

For circular currents we get

$$\begin{gathered} j_\varphi^\omega = -j_1^\omega \sin(\varphi) + j_2^\omega \cos(\varphi) = \\ = \sigma(-E_2^\omega \sin(\varphi) - E_1^\omega \cos(\varphi)) = 0. \end{gathered} \tag{63}$$

Thus these solutions could not change the existence of a stable localized object with rotational symmetry in QHE medium. Only radial currents and circular component of electrical field exist for rotational symmetry superposition of plane solutions with the ($\omega^2 = k^2$) dispersion relation.

2.2.3 Time evolution

The plane wave solutions give the basis for decomposition for any time dependent solution of Maxwell equations in QHE-medium with zero free field terms. Any solution of effective Maxwell equations in Case I is

$$\begin{gathered} \mid \Psi(t,x_1,x_2)\rangle = \mid \Psi^s(x_1,x_2)\rangle + \\ + \mid \Psi_+^\omega(t,x_1,x_2)\rangle + \mid \Psi_-^\omega(t,x_1,x_2)\rangle = \\ = \mid \Psi^s(x_1,x_2)\rangle + \\ + \int dk_1 dk_2 \exp(-ik_1x_1 - ik_2x_2) * \\ \{\exp(ikt) \begin{pmatrix} -\frac{k_2}{k} \\ \frac{k_1}{k} \\ 1 \end{pmatrix} B_+(k_1,k_2) + \\ + \exp(-ikt) \begin{pmatrix} \frac{k_2}{k} \\ -\frac{k_1}{k} \\ 1 \end{pmatrix} B_-(k_1,k_2)\} \end{gathered} \tag{64}$$

Now we consider the Case I when the influence of polaritation currents is taken into account. We try to discuss how the polarization currents can change the properties of solutions of Maxwell equations in QHE medium.

For rotational symmetry superposition of plane waves with polarization currents, the circular part of currents is nonzero

$$j^p_\varphi = -j^p_1 \sin(\varphi) + j^p_2 \cos(\varphi) = \tag{65}$$
$$\sigma(-(E^p_2 + \frac{1}{\omega_c}\partial_t E^p_1)\sin(\varphi) + (-E^p_1 + \frac{1}{\omega_c}\partial_t E^p_2)\cos(\varphi)).$$

Finally, we get

$$j^p_\varphi(r) = \frac{\partial I(r)}{\partial r}, \tag{66}$$

where in the Fourier space

$$I(t,k) = \exp(i\omega_p(k)t)\frac{\sigma}{d(k)}(\Lambda(k) - \frac{\omega_p^2(k)}{\omega_c}(1 + \frac{\beta\Lambda(k)}{\omega_c}))B^p(k) \tag{67}$$

and

$$I(t,r) = \int_0^\infty kdkJ_0(kr)I(t,k).$$

2.3 Polarization currents

The Lorentz symmetry, as we have seen, depends crucially on the absence of diagonal components in the conductivity tensor in QHE. The exactness of this symmetry, therefore, is limited by the "degree of vanishing" of σ_{xx}. Polarization currents, appearing at finite frequencies, may destroy the Lorentz invariance [13, 16].

Even in the plateau regime, where the real part of σ_{xx}vanishes, one should be aware of an imaginary contribution to the diagonal conductivity, caused by the polarization currents , which are proportional to the applied frequency of $E(t)$. At frequencies approaching the cyclotron frequency, this imaginary contribution to σ_{xx} starts to be comparable with the value of the Hall conductivity σ_{xy}:

$$\sigma_{xx} = i\omega\frac{nmc^2}{H^2} = i\frac{\omega}{\omega_c}\sigma_{xy}. \tag{68}$$

The physical origin of Im $\sigma_{xx,}$can be understood as follows: In crossed a.c. electric and d.c. magnetic fields the charge carriers experience an acceleration along the electric field which is proportional to the frequency:

$$\frac{d\mathbf{v}}{dt} = -i\omega c\frac{[\mathbf{E}\times\mathbf{H}]}{H^2} \tag{69}$$

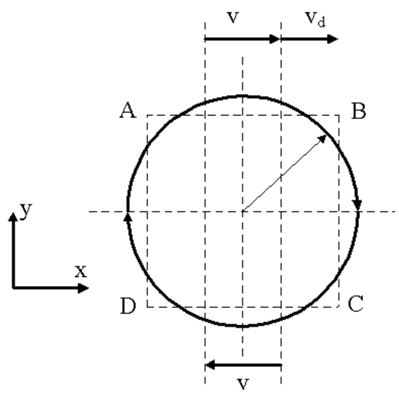

Figure 3: In crossed a.c. electric and d.c. magnetic fields the charge carriers experience an acceleration along the electric field. The corresponding inertial force: $\mathbf{F}_{in} = m d\mathbf{v}/dt$ causes an additional drift velocity: $\mathbf{v}_p = c[\mathbf{F}_{in} \times \mathbf{H}]/eH^2$ and a corresponding current: $j_p = nev_p = nmc^2E/H^2$ in the direction of the time varying electric field $\mathbf{E}(t)$. This contributes to the diagonal conductivity.

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Here

$$V_{dr} \propto \frac{eE_0}{m}\int_A^B e^{i\omega t}dt \propto i\frac{\omega}{\omega_c^2}\frac{eE_0}{m} \tag{70}$$

which yields Eq. (68). Note that σ_{xx} does not depend on the relaxation time τ and remains finite also in the plateau region, in contrast with the vanishing diagonal d.c. conductivity. This new term obviously destroys the Lorentz invariance.

As it was outlined above, the polarization drift under the QHE conditions can be studied using helicon wave propagation in a superlattice with the 2D electron gas in the plateau regime [13]. Experiments of this kind may yield additional information on the exactness of the symmetries discussed above.

2.3.1 Inclusion of the polarization currents

The effective Maxwell equations (40)–(43) with zero free-part E^0 and B^0 in QHE medium with polarization currents [13, 16], as described above, become

$$-ik_1E_1 - ik_2E_2 = 2\pi f(k) j_0(k_1, k_2); \tag{71}$$

$$-i\omega E_1 - ik_2B_3 = 2\pi f(k)\sigma(E_2 + i\frac{\omega}{\omega_c}E_1); \tag{72}$$

$$-i\omega E_2 + ik_1B_3 = 2\pi f(k)\sigma(-E_1 + i\frac{\omega}{\omega_c}E_2); \tag{73}$$

$$-ik_2E_1 + ik_1E_2 - i\omega B_3 = 0. \tag{74}$$

The equations (72)–(74) have nontrivial solutions if

$$\mathrm{Det}\begin{vmatrix} \omega(1+\frac{\beta f_1(k)}{\omega_c}) & -i\beta f(k) & k_2 \\ i\beta f(k) & \omega(1+\frac{\beta f_1(k)}{\omega_c}) & -k_1 \\ k_2 & -k_1 & \omega \end{vmatrix} = 0. \tag{75}$$

The algebraic equation $-\omega\Lambda(\beta/\omega_c)(\Lambda^2+\Lambda\xi-\omega^2) = 0$, where $\Lambda = \sqrt{k^2-\omega^2}$, $\xi = \frac{(1+\beta^2)\omega_c}{\beta} - \frac{\omega^2\beta}{\omega_c}$, has the following roots:

1) $\omega = 0$, which gives the static solutions;

2) $\omega^2 = k^2$, which gives the trivial plane wave solutions for the transverse waves propagating with the light velocity c even in the medium with polarization currents;

3) a new type of solutions arises when $\Lambda = -\frac{\xi}{2}(1-\sqrt{1+\frac{4\omega^2}{\xi^2}})$.

In the region where $(\omega/\omega_c) << 1$ in first approximation we get $\Lambda_p = \beta\frac{\omega^2}{\omega_c(1+\beta^2)}$. This solution gives the following dispersion relation: $\omega^2 = k^2(1-k^2(\frac{\beta}{\omega_c(1+\beta^2)})^2)$ or

$$\omega = \pm\omega_p = \pm k(1-\frac{k^2}{2}(\frac{\beta}{\omega_c(1+\beta^2)})^2). \tag{76}$$

The wave polarization is: $\overrightarrow{E} \perp \overrightarrow{B}$; $\overrightarrow{B} \perp \overrightarrow{k}$; $\cos(\overrightarrow{E} \wedge \overrightarrow{k}) = k\beta\Lambda\frac{1}{d}$.

2.3.2 Evolution of rotational symmetry fluctuation

Let at the initial moment $t = 0$ the local fluctuation of circular currents $j^p_\varphi = \delta j^p_\varphi(r)$ and corresponding fluctuation of magnetic field $B^p(r)$ appear. Then in the presence of polarization currents the dispersion is the reason for spreading of fluctuation at large time. For any smooth localized perturbation with finite energy

$$\mathcal{E}_n = \frac{1}{2}\iint (E_1^2 + E_2^2 + B_3^2)dx_1dx_2$$

satisfying the condition

$$\iint B_\pm^{p2}(x_1, x_2)dk_1dk_2 < \infty. \tag{77}$$

It is well known that any function such that condition (77) is valid can be approximated with any accuracy by a finite sum of Gauss packets [30].

The time behavior for initial Gauss perturbation can be calculated directly. Using this method by direct evaluation, it is easy to show that at very large time when $t \to \infty$ the time depended part of fluctuation goes to 0 on any finite part of the plane at least as $O(1/t^2)$ (for dispersion relation $\omega^2 \simeq k^2$). So the existence of polarization currents changes the picture of existence of stable objects in QHE medium.

If, for example,

$$B^p(t = 0, r) = \frac{b}{2\pi a^2}\exp(-r^2/2a^2) \tag{78}$$

then at the moment t one gets

$$B^p(t, r) = 2\pi \int_0^\infty k\cos(kt)J_0(kr)b\exp(-k^2a^2/2)dk. \tag{79}$$

3 Dimensionality and irreversibility

3.1 Adiabatic invariants

While the quantum motion of an electron in a magnetic field is described by notion of Landau levels, we will remind here some of the basic connections between *adiabatic invariants* in classical mechanics and *energy* levels in quantum physics.

Since the transverse motion of a charge particle in a constant magnetic field is periodic, the associated action integral [31–33]

$$J_{\perp} = \frac{1}{2\pi} \oint \mathbf{P}_{g\perp} d\mathbf{l} \tag{80}$$

(where $\mathbf{P}_g$ is the canonical momentum and $d\mathbf{l}$ a directed line element along the circular path, and integration is over a complete circle) is a constant of motion and, most importantly, an *adiabatic invariant*, that remains constant under *slow* variation of external parameters, such as magnetic field of an RF wave with a frequency lower than the cyclotron frequency of electrons $\omega \ll \omega_c$. Evaluation of the integral in Eq. (80) results in the relation

$$J_{\perp} = \frac{mv_{\perp 0}^2}{2\omega_c} = \frac{E_{\perp}(t)}{\omega_c(t)} \simeq \mathrm{const}(t). \tag{81}$$

Here $\mathrm{E}_{\perp}$ is the kinetic energy of the motion in xy-plane.

For what follows it is useful to connect $J_{\perp}$ with the following physical properties. *The Angular Momentum*:

$$L_z = [\mathbf{r} \times \mathbf{P}_g]_z = -\frac{mV_{\perp 0}^2}{2\omega_c} - J_{\perp}\mathrm{sign}\omega_c$$

is a constant of motion and also an *adiabatic invariant*. In the case of electron motion in a constant magnetic field aligned along the $+z$ direction, $\omega_c < 0$ and $L_z = \frac{mV_{\perp 0}^2}{2|\omega_c|} = J_{\perp}$. *The Orbital Magnetic Moment*:

$$\mu_z = \left(-\frac{e}{m}\right) \frac{mV_{\perp 0}^2}{2\omega_c} = -\frac{e}{m} J_{\perp}\mathrm{sign}\omega_c = \frac{e}{m} L_z$$

is also a constant of motion and an *adiabatic invariant*.

3.1.1 Quasiclassical quantization

The motion of a charge in a magnetic field is periodic in the plane perpendicular to the field and, hence, can be quantized by using the standard Bohr-Sommerfeld quantization condition which after the Peierls substitution $\mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\mathbf{A}$ yields [34, 35]

$$\oint \left(m\mathbf{v} - \frac{e}{c}\mathbf{A}\right) \cdot d\mathbf{r} = (n + \gamma)h \tag{82}$$

where the vector potential $\mathbf{A}$ is related to the magnetic field by

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{83}$$

We can take $\gamma = \frac{1}{2}$ as is the case in the original Bohr-Sommerfeld quantization rule, but its real value follows from an exact solution of the Schrödinger equation.

To calculate the integrals in Eq. (82), we take into account that in the plane perpendicular to the magnetic field the classical orbit of a charged particle is a circumference of the Larmor radius ρ_L along which the particle moves with the velocity $\mid \mathbf{v} \mid = \omega_c \, \rho_L$. Then

$$\oint \mathbf{A} \cdot d\mathbf{r} = \int [\nabla \times \mathbf{A}] \cdot \hat{n} ds = \pi \rho_L^2 B, \oint m\mathbf{v} \cdot d\mathbf{r} = 2\pi \rho_L^2 \frac{eB}{c}.$$

It is easy to see that the Larmor radius $\rho_{L,n}$ is quantized, i.e. takes a discrete set of values depending on the integer n :

$$\rho_{L,n} = \sqrt{\frac{2\hbar}{m\Omega}\left(n + \frac{1}{2}\right)}. \tag{84}$$

In the quasiclassical limit under consideration (i.e., for $n \gg 1$), the Larmor orbit size depends both on the magnetic field strength and the quantum number n. This dependence is given by the relation $\rho_{L,n} \propto \sqrt{\frac{n}{B}}$ .Formally, small values of the quantum number n are beyond the scope of the quasi-classical approximation. On the other hand, it is known that in the case of harmonic oscillator the Bohr-Sommerfeld quantization rule gives an exact formula for the energy spectrum. At $n = 0$ (i.e. in the extreme quantum limit) the Larmor orbit radius becomes equal to:

$$\rho_{L,0} \; = \; \sqrt{\frac{c\hbar}{eB}} \equiv \sqrt{\frac{\Phi_0}{2\pi B}}.$$

In this consideration two fundamental quantities have appeared: the magnetic flux quantum $\Phi_0 = hc/e$ which depends only on the world constants and the magnetic length $L_H = \sqrt{\frac{c\hbar}{eB}}$.

Let us turn to our problem of the energy spectrum calculation.Consider the above quantization in the momentum p-space. The classical equation of motion yields

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\left[\mathbf{B} \times \frac{d\mathbf{r}}{dt}\right]. \tag{85}$$

One can see from this equation that: (i) the electron orbit in the p-space is similar to that in the real space (in $x - y$ plane, when field is directed along the $z-$axis, (ii) in the p-space the orbit is scaled by the factor eB/c, and (iii) the orbit is rotated by $\phi = \pi/2$. Integrating the above equation of motion we obtain:

$$p_n \;=\; \frac{eB}{c}\rho_{L,n} \;=\; \sqrt{2m\hbar\omega_c\left(n+\frac{1}{2}\right)}.$$

Quantization of the orbit radius means also quantization of the momentum p which, in turn, implies quantization of the kinetic energy of the particle. For the quadratic dispersion relation

$$E_n(p_z) = \;\hbar\omega_c\left(n+\frac{1}{2}\right)+\frac{p_z^2}{2m}. \qquad (86)$$

The above quantization rule can be easily generalized to the case of an arbitrary electron dispersion which is the usual case in the crystal solids like metals and semiconductors. The quantization of the $S_p(E)$ is known in the literature as the Lifshitz-Onsager quantization rule. The Lifshitz-Onsager quantization rule is a direct consequence of the commutation rules between the momentum components $\hat{p}_\alpha = (\hbar/i)\partial/\partial x_\alpha + (e/c)A_\alpha$ in the external magnetic field B directed along the z-axes of the Cartesian coordinate system:

$$[\hat{p}_x, \hat{p}_y] = \frac{e\hbar}{c}B, [\hat{p}_y, \hat{p}_z] = [\hat{p}_x, \hat{p}_z] = 0 \qquad (87)$$

(where A_α is the vector-potential). These equations mean that the momentum $\hat{p}_x$ and the coordinate $\hat{q}_x = c\hat{p}_y/eB$ satisfy the commutation rule $[\hat{p}_x, \hat{q}_x] = \hbar/i$ so that the quasiclassical quantization rule holds

$$\oint p_x dq_x = 2\pi\hbar(n+\gamma). \qquad (88)$$

It follows also from the Lifshitz-Onsager quantization rule that the integral $\oint p_x dp_y = S_p(E, p_z)$ equals to the cross-section of the Fermi surface by the plane $p_z = \text{const}$:

$$S(E, p_z) = \frac{2\pi\hbar eB}{c}(n+\gamma). \qquad (89)$$

This is used intensively in the de Haas–van Alphen studies of the Fermi surfaces in metals.

3.2 Alfven cycle

One of the most instructive applications of adiabatic invariance of the orbital magnetic moment of an electron in a magnetic field is the Alfven mechanism

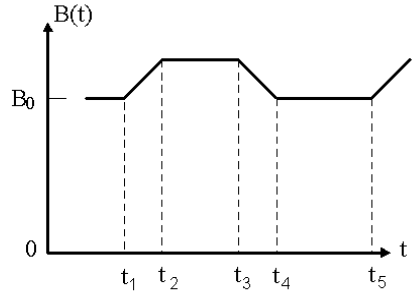

Figure 4: The magnetic field variation $B(t)$ corresponding to the Basic Irreversible Cycle (BIC). Here $\delta \equiv t_2 - t_1 = t_4 - t_3$ and $\Delta \equiv t_3 - t_2 = t_5 - t_4$. It is assumed that $\frac{2\pi}{\omega_c} \ll \delta \ll \tau \ll \Delta$, where τ is the *time* for the energy *equipartition* between $\mathcal{E}_{x,y}$ and $\mathcal{E}_z$.

of the energy transfer from varying in time external magnetic field to ensemble of electrons. This mechanism was studied in connection with acceleration of cosmic particles (*magnetic pumping* [32]), and in solid state (*magnetoviscous damping of helicons and spin waves* [36]). We shall illustrate here the main ideas of magnetic pumping mechanism using a schematic construction in the momentum space of electrons under periodically varying classically strong ($\omega_c \tau \gg 1$) magnetic field where τ is a relaxation time.

Consider the following form of a magnetic field variation (Fig. 4).

$$B(t) = B_0[1 + b(t)],\ b(t) \ll 1 \tag{90}$$

with a period defined by

$$\frac{db}{dt} = \left| \begin{array}{lll} \alpha & \text{if} & t_1 < t < t_2 \\ 0 & \text{if} & t_2 < t < t_3 \\ -\alpha & \text{if} & t_3 < t < t_4 \\ 0 & \text{if} & t_4 < t < t_5 \end{array} \right| \tag{91}$$

here $\alpha \equiv db/dt$. As it was outlined above, an electron in a homogeneous magnetic field possesses an adiabatic invariant:

$$I = \frac{\mathcal{E}_\perp(t)}{B(t)} \simeq \text{const}(t) \tag{92}$$

where $\mathcal{E}_\perp$ is the kinetic energy of the motion in xy-plane. It follows from Eq. (92) that $\mathcal{E}_\perp(t)$ is proportional to $B(t)$, i.e. a periodic variation in $B(t)$

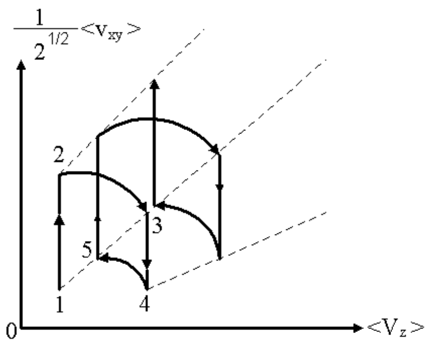

Figure 5: During the time interval $t_1 \Rightarrow t_2$ the field pumps the energy into $\mathcal{E}_\perp$ faster than the collisions couple the oscillatory and translation degrees of freedom (the vertical line). During the time interval $t_2 \Rightarrow t_3$ the collisions restore the equipartition, i.e. some part of the energy is transferred to $\mathcal{E}_z$, and the system returns to equilibrium, point t_3. During the time interval $t_3 \Rightarrow t_4$ only part of the energy, stored in $\mathcal{E}_\perp$ while the field was growing, is returned to the field.

yields a periodic variation in $\mathcal{E}_\perp(t)$ provided the frequency ω of the applied field is smaller than the cyclotron frequency. Since the kinetic energy along the field $\mathcal{E}_z$ is field independent, the total energy variation of a *single electron* over a period of the field variation average to zero

$$\langle \Delta \mathcal{E}_\perp(t) \rangle_T = 0 \tag{93}$$

In an *electron ensemble*, however, due to the equipartition (the energies, stored in perpendicular and parallel to the field electron motion, are periodically restored due to collisions) there is a net energy transfer from the varying in time magnetic field to the electrons. This can be demonstrated in a following way.

Let us denote:$\delta \equiv t_2 - t_1 = t_4 - t_3$ and $\Delta \equiv t_3 - t_2 = t_5 - t_4$. We assume that $\frac{2\pi}{\omega_c} \ll \delta \ll \tau \ll \Delta$, where τ is the *time* for the energy *equipartition* between $\mathcal{E}_{x,y}$ and $\mathcal{E}_z$.

These inequalities and Fig. 4 have the following physical meaning. The field is varying in time fast enough: $\delta \ll \tau$, so that collisions are not efficient in this time interval. Therefore the translational (along the z- axis) motion is decoupled from the oscillatory (in xy-plane) degrees of freedom during the time intervals $t_1 \Rightarrow t_2$ and $t_3 \Rightarrow t_4$. The inequality $2\pi/\omega_c \ll \delta$ assures the adiabatic invariance of the orbital moment. Therefore during these time intervals $\mathcal{E}_{\perp}(\mathrm{t})$ is varying proportionally to the field variations $H(t)$ while $\mathcal{E}_z$ is constant. The inequality $\Delta \gg \tau$ guarantees that the collisions will be effective and the equipartition in the system will be restored during the time intervals $t_2 \Rightarrow t_3$ and $t_4 \Rightarrow t_5$.

3.3 Magnetic billiard with dissipation

In [33, 37] it is shown that such a process in the momentum space of the electron ensemble is irreversible. Indeed, the initial energy gain, 'hidden' in the parallel motion, will be redistributed, finally, between $\mathcal{E}_{\perp}$ and $\mathcal{E}_z$ during the time interval $t_4 \Rightarrow t_5$. Therefore, the temperature of the electron ensemble will grow: $T(t_5) > T(t_1)$. This is a typical irreversible process, resulting in the magnetoviscosity of the three-dimensional electron gas under a strong magnetic field.

The two-dimensional limit corresponds to the trajectory $1 \Rightarrow 2 \Rightarrow 1$, and the energy of a 2DEG at the end of the cycle equals to its energy at the beginning.

In two-dimensional electron system, therefore, we encounter a paradox: it turns to be reversible under a periodic external force. This obviously contradicts the basic laws of statistical physics: no macroscopic system is reversible [39].

The two-dimensional electron sytem under external magnetic field is a macroscopic, however reversible system independent whether the quantum mechanical limit, well separated Landau levels, is achieved.

4 Helicon waves

4.1 Isotropic model

Let us turn now to possible condensed matter realisations of these general ideas using the electromagnetic mode which has most of it energy in the magnetic field wave component: the *helicons* [33, 40–45].

Because of a very high refractive index, the group velocity of the helicon is extremely slow and most of the wave energy is stored in its magnetic component. This opens a unique possibility for studying the responce of a dense electron plasmas to temporal and spatial variations of magnetic fields.

Qualitatively, the helicon propagation can be described as follows [44]. When the mean free path of electrons is sufficiently large and the frequency ω of the wave is low enough, the electrons affected by the Lorentz force

$-|e|\,[\mathbf{v}\times\mathbf{B}_0]$ would drift in the direction perpendicular to the plane formed by the uniform magnetic field $\mathbf{B}_0$ and the electric field $\mathbf{E}$ of the wave. The current $\mathbf{j}$ created by such a drift is in fact a Hall current. It is perpendicular to the electric field $\mathbf{E}$ of the wave and causes no dissipation. Consequently, the electromagnetic energy is conserved and in the absence of collisions the wave does not attenuate. The transverse current that generate a time dependent magnetic field is sufficient to maintain self-sustaining oscillations. It is essentially an *RF Hall effect*. The helicon is therefore a low-lying circularly polarized magnetic excitation in an electron system placed under classically strong magnetic field.

In the collisionless limiting case $\tau\to\infty$ the helicon dispersion relation is [44]: $\omega=\frac{\omega_c c^2}{\omega_p^2}k^2$ (ω_p is the plasma frequency) and its phase velocity is very low compared to the light velocity: $v_{\mathrm{ph}}^H=c\frac{\sqrt{\omega\omega_c}}{\omega_p}$. The helicon frequencies in metals [42] and doped semiconductors are small compared to the cyclotron and plasma frequencies ω_c and ω_p. While in vacuum a usual electromagnetic wave has a magnetic field $B\sim E/c$, in a helicon wave $B_H\sim E/v_{\mathrm{ph}}^H\gg E/c$. Hence, the field of the helicon wave is mainly magnetic.

4.2 Non-locality and temporal dispersion

A rich physics is contained in peculiarities of the helicon dispersion and damping due to the non-locality effects [44] and temporal dispersion [33, 36]. Two limiting cases ($\omega\tau\gg1$ and $\omega\tau\ll1$) of the helicon wave propagation are well understood and calculated in the framework of the kinetic theory.

a) $\omega\tau\ll1$: the local non-dispersive collisional damping [41]: $D\propto(\omega_c\tau_p)^{-1}$. This is just a collisional damping due to elastic scattering of electrons.

b) $\omega\tau\gg1$: the non-local magnetic Landau damping [43, 44], which is characterised by a linear in k damping factor: $D=\frac{Im\omega}{Re\omega}\propto kR_L\sin\Delta$, R_L is the Larmor radius, τ_p the momentum relaxation time.

Its physical meaning can be understood as follows [44]. In the case of the oblique propagation the helicon wave is not purely transverse, i.e. the external magnetic field is modulated by z-component of the wave magnetic field, which results in two specific mechanisms of damping. The *standard Landau damping* [31] is due to the small longitudinal component of the wave electric field and is negligibly small since the helicon phase velocity is much less than the Fermi velocity of carriers. The essential damping of obliquely propagating helicons is connected with a considerable magnetic field of the wave. The variable magnetic field $\mathbf{B}_H$ being added to the static magnetic field $\mathbf{B}_0$ forms a periodic magnetic field acting on the electron magnetic moment $\mu_m = \dfrac{mv_\perp^2}{2B_0}$ with a force: $F_H = e\,[\mathbf{v} \times \mathbf{B}]_z \simeq \mu_m \dfrac{\partial B_z}{\partial z}$. As a result, the gyrating particle is interacting with the variable magnetic field B_{Hz} of the helicon which periodically adds to and subtracts from the static magnetic field $\mathbf{B}_0$.This interaction is especially strong when the velocity of the particle is close to the velocity of the moving magnetic mirror along the Z direction $v_z \simeq \frac{\omega}{k_z}$.

Similar to the standart *Landau damping*, the particles with the velocity just less than ω/k_z will be accelerated by the magnetic mirror and will extract the energy from the wave. The particles with v_z slightly greater than ω/k_z will be slowed down. Since in thermal equilibrium $\partial f_0/\partial v < 0$, there are more slowly moving particles than the fast ones. Consequently, the interaction will result in the *magnetic Landau damping* of the helicon wave.

c) In the intermediate region ($\omega\tau \simeq 1$), the damping factor has a Lorenzian shape, centred at $\omega \simeq \tau_p$ [36]:

$$D \simeq \left(\frac{n_o e_F}{H_o^2}\right)(\omega\tau_p)\left((1+\omega\tau_p)^2\right)^{-1} \quad (\omega\tau_p \simeq 1) \tag{94}$$

where n_o is the electron density, e_F is the Fermi energy, H_o is the external magnetic field. This damping mechanism is the manifestation of the *magnetoviscous damping* [36], which can be understood as follows.

While the time dependent energy variation of a single electron is averaged to zero, due to the adiabatic invariance of its classical orbital moment, an ensemble of 3D electrons will gain the energy stored in the wave, due to the energy transfer from the time-dependent part of the magnetic field to the electron subsystem, causing the magnetoviscosity described above. This mechanism is most effective [32, 36] when the frequency of the field is comparable with the momentum collisional frequency of the electrons.

The magnetic Landau damping and the magnetoviscous damping of an electromagnetic wave is caused by two different groups of electrons. Therefore, for metals with a non-spherical Fermi surface, in a geometry with the 'resonant' electrons occupying the regions on the Fermi surface with low density of states, the magnetoviscous damping may prevail over the magnetic Landau damping.

4.3 Quantum Hall helicon mode

Let us discuss now the helicon wave propagation in the extreme anisotropic case, i.e. the case of two-dimensional electron layers under strong magnetic field [13]. These could be semiconductor superlattices or multi-quantum wells, where the conventional QHE [1] caused by the pinning of the chemical potential by impurities in the energy gap between two adjacent Landau levels, can occur. Another interesting possibility could be the dense electron superlattices where the diamagnetic phase transition may result in the Ideally Conducting Phases (ICP), with the trapping of the chemical potential within the Landau gaps resulting in lowering of the total energy of the 2DEG system [46].

Experimentally, in the GaAs/AlGaAs heterostructure in magnetic fields of about 8T, at a temperature of 1.23K, the upper limit on the electron scattering time is estimated as 1.5x10^{-3}sec. This means that the high frequency ($\omega\tau > 1$) regime, which in conventional semiconductors starts only in the far infrared, may start at kilohertz frequencies in the QHE.

Under the quantum Hall effect conditions, assuming zero conductance in z-direction: $\sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = \sigma_{zz} = 0$ and isotropy in the other two directions: $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$, we are left with only two independent components of the conductivity tensor: σ_{xy} and σ_{xx}. Assuming a model superlattice with 2DEG layers of width a, separated by insulating layers of width b with a dielectric constant ϵ_0,the wave equation can be reduced to the form [13]

$$E^+ + (\frac{\omega}{c})^2 \epsilon_- E^+ = 0 \tag{95}$$

where $E^+ = E_x + iE_y$ and $\epsilon = \epsilon_{xx} - i\epsilon_{xy}$. A standard procedure then yields the Penney-Kronig like dispersion relation.

Assuming now that Re $\sigma_{xy}^{2d} \simeq (n_s ec)/H$; Im $\sigma_{xy} \simeq$ Re $\sigma_{xx} \simeq 0$; Im $\sigma_{xx} \simeq \sigma_{xy}$ (ω/ω_c), we obtain, in the frequency regime:

$$\omega < \omega_c < \omega_p : k_1^2 = \frac{\omega}{c^2} 4\pi\sigma_{xy}^{3d} \left(1 + \frac{\omega}{\omega_c}\right)$$

which is the standard 3D-helicon dispersion with $\sigma_{xy}^{3d} = \frac{1}{d}\sigma_{xy}^{2d}$.

The obtained transcendental dispersion equation can be reduced, in the long wave-length limit $kd \ll 1$, to a simple algebraic one:

$$\omega^2 \epsilon_o + 4\pi\alpha\sigma_{xy}\omega \left(1 + \frac{\omega}{\omega_c}\right) - c^2 k^2 = 0 \tag{96}$$

where $\alpha = a/d < 1$. At very low frequencies, the term $\frac{\omega}{\omega_c}$ in brackets can be neglected, and one gets the following, helicon-like, solution:

$$\omega = \frac{c^2 k^2}{4\pi\alpha\sigma_{xy}} = \frac{c^2 k^2 h}{\alpha n_F e^2}.$$

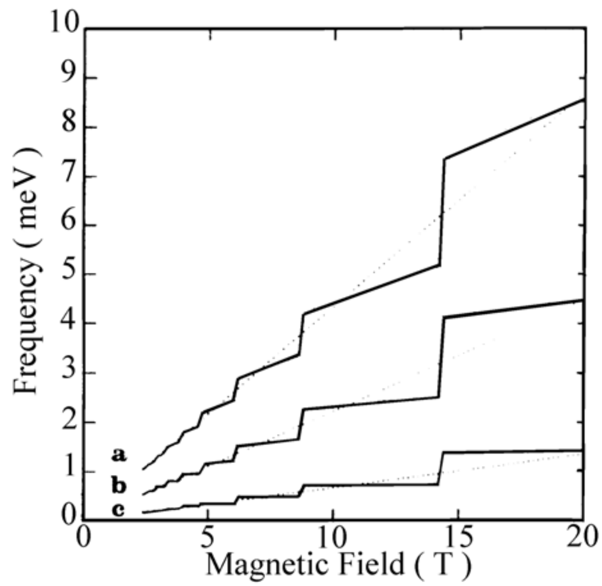

Figure 6: Magnetic field dependence of a quantum helicon mode in a multi-quantum-well structure with the electrons in a QHE state.

The second equality assumes that we are in the QHE regime, namely, that there are n_F completely filled Landau levels and that the Fermi energy is in a magnetic energy gap.

In Fig. 6, the numerical solutions are presented for a GaAs/$Al_x Ga_{1-x}As$ supelattice wth the following parameters: the electron concentration is $n_o = 5 \times 10^{11} cm^{-2}$, the GaAs layers are 100 $\mathring{A}$ thick, the $Al_x Ga_{1-x}As$ barriers are 400 $\mathring{A}$ thick. The number of 2DEG layers in a superlattice: a) N=300; b) N=500; c) N=1000 [13].

At higher frequencies the plateaus are distorted. This is caused by the usual displacement currents and by the *polarization currents*, arising from the nonvanishing imaginary part of diagonal conductivity [16], as it was discussed earlier.

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