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On the problem of *Zitterbewegung* of the Dirac electron *

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Abstract

Reformulation of Dirac equation in terms of real quadratic division algebra of quaternions is given. Similar equations with different mass term are identified as suitable for description of free propagating quark motion. The complete orthonormal set of the positive-energy plane wave solutions is presented. Therefore, *Zitterbewegung* phenomenon is absent in this formulation. The probability current is proportional to the momentum, as in standard Schrödinger wave mechanics.

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1 Introduction

In this paper I will discuss an old, well-known and apparently "insignificant" problem that lies in the foundation of the relativistic quantum mechanics or, more precisely, in the foundation of the quantum electrodynamics. A story starts with the celebrated paper of P.A.M. Dirac [1] that describes relativistic motion of a free propagating electron, and had tremendous success of natural explanation of spin, correct non-relativistic limit, correct coupling with the external magnetic field, correct gyromagnetic ratio and, finally, prediction of positron.

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The mathematical formalism used requires presence of the negative kinetic energy states in order to obtain the complete orthonormal set of the linearly independent fundamental solutions of the suggested equation. Indeed, such states do not make sense from the physical point of view.

Since the Dirac equation is written in the Hamiltonian form, it allows us to work in the Heisenberg representation and determine directly whether or not a given observable is a constant of motion. Then we find, for example, that the momentum is a conserved quantity as it should be. However, the orbital momentum, as well as the spin, are not conserved separately and only the sum of them is a constant of motion. Conventionally, the spin is associated with the internal degree of freedom of the electron and therefore apparently has nothing to do with isotropy of the space-time continuum (indeed, if we assign to the quantum mechanical space-time point internal algebraic structure, then this will be rather naturally expected result). Even more surprising result [2] is obtained if we consider velocity $\dot{\vec{x}}$ in the Dirac formulation. Instantaneous group velocity of the electron has only values $\pm c$ in spite of the non-zero rest mass of electron. In addition, velocity of a free moving electron is not a constant of motion.

An analytic solution for the coordinate operator of a free propagating electron was found by E. Schrödinger [3]. It turns out that in addition to the uniform rectilinear motion consistent with the classical electrodynamics, the Dirac electron executes oscillatory motion, which E. Schrödinger called *Zitterbewegung*. Let us recall that entire non-relativistic quantum mechanics was raised in order to explain the absence of radiation during the oscillatory motion of the electron bounded by the electric potential of the nucleus. Therefore, the Dirac theory of electron contains a definite prediction that the free moving electron will loose all a kinetic energy through electromagnetic radiation [4].

It is rather surprising that the *Zitterbewegung Problem* attracted only sporadic [5, 6] attention during years of development of the theory of quantum fields and efforts to achieve the unification of all fundamental interactions. It was demonstrated [6, 7] that the *Zitterbewegung* oscillations are due solely to interference between the positive- and negative- energy components in the wave packet. The *Zitterbewegung* is completely absent for a wave packet made up exclusively of positive energy plane wave solutions. It is clear from the above analysis that if one achieves the reformulation of the Dirac equation such that the complete orthonormal set of linearly independent solutions will contain only positive energy states then the *Zitterbewegung* oscillations will disappear. Indeed, the charge-conjugated solutions, associated with the positron, must be retained.

It has been known for a long time that the algebraic structure of Dirac equation is closely related to the quadratic normal division algebra of quaternions. Here we suggest a quaternionic reformulation of the Dirac equations [8], as well as an additional set of similar equations suitable for description of the free propagating quark motion. The main effort is made to obtain equations with the intrinsic $SU(2) \otimes U(1)$ local gauge invariance. In contrast with the approach of S.L. Adler and others [9, 10], we consider the possibility that the previously obtained quaternionic extension [11] of the Hilbert space description of quantum fields represents a consistent mathematical framework for the electroweak unification scheme (a brief summary of relevant results is given in the Appendix). It is obvious that in order to achieve unification of all fundamental interactions, the algebraic extension beyond the quaternions is needed. We demonstrate that mathematical structure of the obtained equations of motion suggests that the required extension may proceed through wave functions which possess three and seven phases, whereas the scalar product remains complex. In that case the examples of nonextendability to octonionic quantum mechanics [9] are not valid.

2 Equations of motion for fundamental fermions

Let us consider the algebraic structure of the Dirac equation. The problem is to achieve factorization of the energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4 \tag{1}$$

in such a way, that the correspondent Hamiltonian is the generator of Abelian translations in time, which is expressed by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{2}$$

It was demonstrated by P.A.M. Dirac [1] that in terms of the two-dimensional commutative quadratic division algebra of complex numbers no solution can be found. The problem requires intrinsically an extension of the algebraic basis of the theory. The Dirac's solution of the problem,

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i}\alpha_j \frac{\partial \psi}{\partial x_j} - \beta\, mc^2\, \psi \equiv H\psi, \qquad j = 1,2,3 \tag{3}$$

uses the generators of the C_4 Clifford algebra:

$$
\begin{array}{ll}
\alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} & \\
\alpha_i \beta + \beta \alpha_i = 0 & i, k = 1, 2, 3 \\
\alpha_i^2 = \beta^2 = 1. &
\end{array}
\tag{4}
$$

However, such a drastic growth in algebra is only apparent. The true physical content of the obtained result is expressed more distinctly if (3) is written in the following form:

$$
\begin{array}{ll}
i\frac{1}{c}\frac{\partial \psi}{\partial t} + i\sigma_j \frac{\partial \psi}{\partial x_j} = \frac{mc}{\hbar}\phi & \\
& j = 1, 2, 3 \\
-i\frac{1}{c}\frac{\partial \phi}{\partial t} + i\sigma_j \frac{\partial \phi}{\partial x_j} = -\frac{mc}{\hbar}\psi &
\end{array}
\tag{5}
$$

(we choose to work in the Weyl representation [12]

$$
\begin{array}{ll}
\alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} & \\
\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & i = 1, 2, 3 \\
\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} &
\end{array}
\tag{6}
$$

by the reason, which will be explained below).

In a precise analogy with the physical content of Maxwell's equations, we again have to deal with two mutually connected waves, which propagate together in a space with constant velocity.

Now, the full basis of the certain algebra (C_2 Clifford algebra) is symmetrically used in (5). However, it is assumed that i in (5) commutes with σ_i, that is the algebra is defined over the field of complex numbers. Therefore, the algebraic foundation of this formulation is based on an eight-dimensional non-division algebra. In addition, it is customary in the applications to continue working with complex numbers as an abstract algebra, but for the C_2 Clifford algebra, one makes use of a representation (Pauli matrices) introducing into the theory an asymmetry, which has neither mathematical nor physical justification.

Now, I will demonstrate that the algebraic foundation of (5) may be reduced to a four-dimensional real quadratic division algebra of quaternions and that the structure of Dirac equations is intrinsically connected with the functional-analytical structures mentioned above.

First of all, let us substitute

$$e_j = -i\sigma_j \qquad j = 1,2,3 \tag{7}$$

into (5). Then

$$\begin{aligned} i\frac{1}{c}\frac{\partial\psi}{\partial t} - e_j\frac{\partial\psi}{\partial x_j} &= \frac{mc}{\hbar}\phi \\ -i\frac{1}{c}\frac{\partial\phi}{\partial t} - e_j\frac{\partial\phi}{\partial x_j} &= -\frac{mc}{\hbar}\psi \end{aligned} \qquad j = 1,2,3 \tag{8}$$

or, equivalently,

$$\begin{aligned} &\left\{\begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix}\frac{\partial}{\partial x_j}\right\}\begin{pmatrix} \psi \\ \psi i \end{pmatrix} \\ &\qquad = \frac{mc}{\hbar}\begin{pmatrix} e_0 & 0 \\ 0 & -e_0 \end{pmatrix}\begin{pmatrix} \phi \\ \phi i \end{pmatrix} \\ &\left\{-\begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix}\frac{\partial}{\partial x_j}\right\}\begin{pmatrix} \phi \\ \phi i \end{pmatrix} \\ &\qquad = -\frac{mc}{\hbar}\begin{pmatrix} e_0 & 0 \\ 0 & -e_0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi i \end{pmatrix} \end{aligned} \qquad j = 1,2,3 \tag{9}$$

and

$$\begin{aligned} &\left\{\begin{pmatrix} 0 & -e_0 \\ e_0 & 0 \end{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & e_j \end{pmatrix}\frac{\partial}{\partial x_j}\right\}\begin{pmatrix} \psi \\ -\psi i \end{pmatrix} \\ &\qquad = \frac{mc}{\hbar}\begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix}\begin{pmatrix} \phi \\ -\phi i \end{pmatrix} \\ &\left\{-\begin{pmatrix} 0 & -e_0 \\ e_0 & 0 \end{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & e_j \end{pmatrix}\frac{\partial}{\partial x_j}\right\}\begin{pmatrix} \phi \\ -\phi i \end{pmatrix} \\ &\qquad = -\frac{mc}{\hbar}\begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix}\begin{pmatrix} \psi \\ -\psi i \end{pmatrix}. \end{aligned} \qquad j = 1,2,3 \tag{10}$$

Notice that the states have the form (A1) and all operators have the form (A11) and (A12).

Besides that, an additional (and only one) mass term is allowed:

$$
\begin{aligned}
&\left\{\begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix} \frac{\partial}{\partial x_j}\right\} \begin{pmatrix} \psi \\ \psi i \end{pmatrix} \\
&\quad = \left\{\frac{m_1 c}{\hbar} \begin{pmatrix} e_0 & 0 \\ 0 & -e_0 \end{pmatrix} + \frac{m_2 c}{\hbar} \begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix}\right\} \begin{pmatrix} \phi \\ \phi i \end{pmatrix} \qquad j = 1,2,3 \\
&\left\{-\begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix} \frac{\partial}{\partial x_j}\right\} \begin{pmatrix} \phi \\ \phi i \end{pmatrix} \\
&\quad = \left\{-\frac{m_1 c}{\hbar} \begin{pmatrix} e_0 & 0 \\ 0 & -e_0 \end{pmatrix} + \frac{m_2 c}{\hbar} \begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix}\right\} \begin{pmatrix} \psi \\ \psi i \end{pmatrix} \\
&\left\{\begin{pmatrix} 0 & -e_0 \\ e_0 & 0 \end{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & e_j \end{pmatrix} \frac{\partial}{\partial x_j}\right\} \begin{pmatrix} \psi \\ -\psi i \end{pmatrix} \\
&\quad = \left\{\frac{m_1 c}{\hbar} \begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix} + \frac{m_2 c}{\hbar} \begin{pmatrix} 0 & -e_0 \\ e_0 & 0 \end{pmatrix}\right\} \begin{pmatrix} \phi \\ -\phi i \end{pmatrix} \qquad j = 1,2,3 \\
&\left\{-\begin{pmatrix} 0 & -e_0 \\ e_0 & 0 \end{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} - \begin{pmatrix} e_j & 0 \\ 0 & e_j \end{pmatrix} \frac{\partial}{\partial x_j}\right\} \begin{pmatrix} \phi \\ -\phi i \end{pmatrix} \\
&\quad = \left\{\frac{m_1 c}{\hbar} \begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix} + \frac{m_2 c}{\hbar} \begin{pmatrix} 0 & -e_0 \\ e_0 & 0 \end{pmatrix}\right\} \begin{pmatrix} \psi \\ -\psi i \end{pmatrix}.
\end{aligned} \tag{11}
$$

It may be verified that the energy-momentum relation is not spoiled if one defines

$$M \equiv \sqrt{m_1^2 + m_2^2}. \tag{12}$$

Here we are forced to consider masses as given phenomenological parameters. If $m_1 \neq 0$, the presence of this additional term does not increase the number of fundamental plane wave solutions of the equations (11). Therefore, we will consider the equations (11) with $m_1 = 0$ as a separate independent set and in order to maintain the direct connection with the Dirac equations, will neglect the m_2 term in the presence of the non-vanishing m_1 term.

Indeed, only two equations (11) are independent:

$$\begin{aligned} \frac{1}{c}\frac{\partial\psi}{\partial t}i \,-\, e_j\frac{\partial\psi}{\partial x_j} \;&=\; \frac{mc}{\hbar}\phi \\ & \qquad\qquad\qquad j = 1,2,3 \\ -\frac{1}{c}\frac{\partial\phi}{\partial t}i \,-\, e_j\frac{\partial\phi}{\partial x_j} \;&=\; -\frac{mc}{\hbar}\psi. \end{aligned} \tag{13}$$

The form (13) is very convenient for the investigation of gauge invariance group of the Dirac equations. The $U\ (1)$ gauge invariance group from the right is generated by the transformations

$$\begin{aligned} &\psi' \;=\; \psi\, z, \quad \phi' \;=\; \phi\, z \\ &z \;=\; a \,+\, bi, \quad |z| \;=\; 1,\ a,\ b \text{ are real numbers.} \end{aligned} \tag{14}$$

Since for every pair of solutions $(\psi,\ \phi)$ of the linear differential equations, $(\psi a,\ \phi a)$ (a is a real number) is also a solution, it is always enough to show that the particular transformation

$$\begin{aligned} \psi' \;&=\; \psi\, i \\ & \qquad\qquad (\text{i.e.}\quad \mathrm{a} \;=\; 0, \qquad \mathrm{b} \;=\; 1) \\ \phi' \;&=\; \phi\, i \end{aligned} \tag{15}$$

leaves the equations invariant.

Invariance of the equations (13) with respect to this transformation is obvious. Let us consider what is a left gauge invariance group of the Dirac equations. Remember that (13) are the equations for free propagating waves and thus admit solutions of the form

$$\begin{aligned} \psi \;&=\; U_1\ \exp\frac{-i\,(Et\ -\ \vec{p}\vec{x})}{\hbar} \\ \phi \;&=\; U_2\ \exp\frac{-i\,(Et\ -\ \vec{p}\vec{x})}{\hbar}. \end{aligned} \tag{16}$$

Therefore, the $U\ (1)$ transformations

$$\begin{aligned} \psi' \;&=\; z_1\psi, \qquad \phi' \;=\; z_1\phi \\ z_1 \;&=\; a \,+\, bi_1, \qquad |z_1| \;=\; 1 \\ i_1 \;&\equiv\; \frac{e_1p_1 \,+\, e_2p_2 \,+\, e_1e_2p_3}{|\vec{p}|} \end{aligned} \tag{17}$$

leave the equations (13) invariant and constitute the left gauge invariance group of the Dirac equations.

In order to see that, it is sufficient to show again that

$$\begin{aligned} \psi' &= i_1 \psi \\ \phi' &= i_1 \phi \end{aligned} \tag{18}$$

is a solution of the equations (13):

$$\frac{i_1 U_1}{c} (-iE)\ i - e_1 i_1 U_1\ (ip_1) - e_2 i_1 U_1\ (ip_2) - e_1 e_2 i_1 U_1\ (ip_3) = mc\ i_1 U_2$$

$$\frac{i_1 U_2}{c} (iE)\ i - e_1 i_1 U_2\ (ip_1) - e_2 i_1 U_2\ (ip_2) - e_1 e_2 i_1 U_2\ (ip_3) = -mc\ i_1 U_1. \tag{19}$$

Then

$$\begin{aligned} &\frac{i_1 U_1 E}{c} - (e_1 p_1 + e_2 p_2 + e_1 e_2 p_3)\ i_1 U_1\, i = mc\ i_1 U_2 \\ &-\frac{i_1 U_2 E}{c} - (e_1 p_1 + e_2 p_2 + e_1 e_2 p_3)\ i_1 U_2\, i = -mc\ i_1 U_1. \end{aligned} \tag{20}$$

By definition (see (17)),

$$e_1 p_1 + e_2 p_2 + e_1 e_2 p_3 = i_1\ |\vec{p}|\,. \tag{21}$$

Therefore,

$$\begin{aligned} i_1\ &\left[\frac{U_1 E}{c} - (e_1 p_1 + e_2 p_2 + e_1 e_2 p_3)\ U_1 i\right] = i_1\ (mc U_2) \\ i_1\ &\left[-\frac{U_2 E}{c} - (e_1 p_1 + e_2 p_2 + e_1 e_2 p_3)\ U_2 i\right] = i_1\ (-mc U_1)\,. \end{aligned} \tag{22}$$

It is assumed in the Dirac equations [13], that

$$[i, e_j] = 0, \qquad j = 1, 2, 3 \tag{23}$$

and hence the obtained gauge invariance group is $U(1) \otimes U(1)$. Now it becomes clear why the Dirac equations allow us to incorporate an additional charge [14] and turn out to be suitable for the realization of the electroweak unification scheme [15] without contradiction with the Aharonov-Bohm effect [16].

However, the group-theoretical content of this scheme [15], side by side with the functional-analytical structures [11], suggests that the left gauge invariance group should be larger (at least $U\ (1;q)\ \cong\ SU\,(2)$) and should not contain an Abelian invariant subgroup. The simplest way to satisfy these requirements is to identify the Abelian groups (14) and (17) discussed above, that is to attach to the Dirac equations the following form:

$$\begin{aligned} \frac{1}{c}\frac{\partial\psi}{\partial t}i - \frac{\partial\psi}{\partial x_j}e_j &= \frac{mc}{\hbar}\phi \\ -\frac{1}{c}\frac{\partial\phi}{\partial t}i - \frac{\partial\phi}{\partial x_j}e_j &= -\frac{mc}{\hbar}\psi \end{aligned} \qquad j = 1,2,3 \tag{24}$$

and to drop the assumption (23). Then the algebraic foundation of the theory is reduced to a four-dimensional real quadratic division algebra of quaternions.

The $U\,(1)$ right gauge invariance of the equations (24) may be maintained if

$$i \ = \ \frac{e_1p_1 \ + \ e_2p_2 \ + \ e_1e_2p_3}{|\vec{p}|} \tag{25}$$

and may be demonstrated exactly in the same way as (18) - (22).

Consequently, we have obtained the additional meaning for i, which appears originally in the Schrödinger equation. An algebra itself forms a vector space, and the basis of algebra constitutes a suitable set of orthogonal axes in that space, for example, a complex algebra may be considered as a two-dimensional plane with the orthogonal directions 1 and i. In that space i standing in the left-hand side of the Schrödinger equation define the direction of the time translations, which form an Abelian group. Therefore, the $U\,(1)$ right gauge invariance of the equations (24) leads us to the conclusion that these equations define (the condition (25)) the direction of the time translations at the three-dimensional quaternionic surface (the space of quantum mechanical phases). Then, a possible physical interpretation is that, compared with a classical relativistic particle, a quantum particle has not only its proper time but, in addition, a proper direction of time. Perhaps, this may serve as an explanation of why quantum equations of motion contain the first time derivative, whereas the classical equations of motion are expressed in terms of the second derivative.

We investigate now how our manipulations have affected the corresponding solutions. As it is well known, the general solution of Dirac equation may be formed as a linear combination of the four independent solutions, which are four spinors with four components. Two of them are obtained for $E > 0$ for two spin states $U_2^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $U_2^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. The other two we are forced to obtain using $E < 0$ since there are no other possibilities. They correspond to the arbitrary choice of $U_1^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $U_1^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let us check what happens in our quaternionic version of the Dirac equation. In order to maintain connection with the original Dirac solutions, let us form a complete orthonormal set of it:

$$\begin{aligned}
\psi_D^{(1)} &= \begin{pmatrix} U_1\ (\vec{p}) \\ U_1\ (\vec{p})\, i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(2)} &= \begin{pmatrix} U_2\ (\vec{p}) \\ U_2\ (\vec{p})\, i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(3)} &= \begin{pmatrix} U_1\ (\vec{p}) \\ -U_1\ (\vec{p})\, i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(4)} &= \begin{pmatrix} U_2\ (\vec{p}) \\ -U_2\ (\vec{p})\, i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar}
\end{aligned} \tag{26}$$

we have

$$\begin{aligned}
\left(\frac{E}{c}\ +\ |\vec{p}|\right)\ U_1\ -\ mc\, U_2\ &=\ 0 \\
-\ mc\, U_1\ +\ \left(\frac{E}{c}\ -\ |\vec{p}|\right)\ U_2\ &=\ 0.
\end{aligned} \tag{27}$$

The existence of non-trivial solutions is ensured by

$$\frac{E^2}{c^2}\ -\ |\vec{p}|^2\ -\ m^2c^2\ =\ 0$$

and

$$U_1^{(1,2)}\ =\ \frac{mc^2}{E\ +\ c\,|\vec{p}|} U_2^{(1,2)}. \tag{28}$$

Let

$$U_2^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } U_2^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\psi_D^{(1)} &= \frac{mc^2 N_1}{E + c\,|\vec{p}|} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
&= \frac{mc^2 N_1}{E + c\,|\vec{p}|} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(2)} &= N_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
&= N_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar}.
\end{aligned} \tag{29}$$

Now

$$U_2^{(3,4)} = \frac{(E + c\,|\vec{p}|)}{mc^2} U_1^{(3,4)}. \tag{30}$$

Let

$$U_1^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } U_1^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\psi_D^{(3)} &= N_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
&= N_2 \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(4)} &= \frac{(E + c\,|\vec{p}|) N_2}{mc^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
&= \frac{(E + c\,|\vec{p}|) N_2}{mc^2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar}
\end{aligned} \tag{31}$$

Our choice is made in order to compare with the standard set of the linearly independent solutions for the Dirac equation. Indeed, the alternative

$$\psi_D^{(3)} = \frac{mc^2 N_1}{E + c\,|\vec{p}|} \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar}$$

and (32)

$$\psi_D^{(4)} = N_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar}.$$

may serve us equally well and at the same time make things more transparent, since we have obtained exactly the same solutions as $\psi_D^{(1)}$ and $\psi_D^{(2)}$, which would be negative energy solutions in the Dirac equation with $\vec{p}' = -\vec{p}$ if we make the substitution

$$i' = -i = -\frac{e_1 p_1 + e_2 p_2 + e_1 e_2 p_3}{|\vec{p}|}. \tag{33}$$

Obviously, the obtained set is mutually orthogonal.

Finally, using standard normalization condition, we obtain:

$$\begin{aligned}
\psi_D^{(1)} &= \frac{1}{2}\sqrt{\frac{mc^2}{E + c\,|\vec{p}|}} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(2)} &= \frac{1}{2}\sqrt{\frac{E + c\,|\vec{p}|}{mc^2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(3)} &= \frac{1}{2}\sqrt{\frac{mc^2}{E + c\,|\vec{p}|}} \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\
\psi_D^{(4)} &= \frac{1}{2}\sqrt{\frac{E + c\,|\vec{p}|}{mc^2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix} \exp \frac{-i\ (Et\ -\ \vec{p}\vec{x})}{\hbar}.
\end{aligned} \tag{34}$$

The obtained solutions maintain symmetry with respect to space coordinates that may be expected based on the assumption of homogeneity of the space-time continuum. Indeed, the correctness of the suggested equations may be verified only through careful comparison with the experimental data.

Now let us consider similar equations

$$\begin{aligned} \frac{1}{c}\,\frac{\partial \psi}{\partial t}\,i \;-\; \frac{\partial \psi}{\partial x_j}\,e_j \;&=\; \frac{mc}{\hbar}\,\phi\,i \\ -\frac{1}{c}\,\frac{\partial \phi}{\partial t}\,i \;-\; \frac{\partial \phi}{\partial x_j}\,e_j \;&=\; \frac{mc}{\hbar}\,\psi\,i \end{aligned} \qquad j = 1, 2, 3 \tag{35}$$

and verify that they admit an additional set of plane wave solutions, for example, in the following form:

$$\begin{aligned} \psi_j \;&=\; U_1 e_j \exp \frac{-e_j\;(Et \;-\; \vec{p}\vec{x})}{\hbar} \\ \phi_j \;&=\; U_2 \exp \frac{-e_j\;(Et \;-\; \vec{p}\vec{x})}{\hbar}\;. \end{aligned} \qquad j = 1, 2, 3 \tag{36}$$

Here U_1 and U_2 are assumed to be real numbers. Then

$$\begin{aligned} U_1 e_j\;\left(-\frac{e_j E i}{c} \;-\; e_j p_1 e_1 \;-\; e_j p_2 e_2 \;-e_j p_3 e_1 e_2\right) \;&=\; mcU_2 i \\ U_2\;\left(\frac{e_j E i}{c} \;-\; e_j p_1 e_1 \;-\; e_j p_2 e_2 \;-e_j p_3 e_1 e_2\right) \;&=\; mcU_1\; e_j i \end{aligned} \tag{37}$$

or

$$\begin{aligned} U_1 e_j\;\left(-\frac{e_j E i}{c} \;-\; e_j\;(e_1 p_1 \;+\; e_2 p_2 \;+\; e_1 e_2 p_3)\right) \;&=\; mcU_2 i \\ U_2\;\left(\frac{e_j E i}{c} \;-\; e_j\;(e_1 p_1 \;+\; e_2 p_2 \;+\; e_1 e_2 p_3)\right) \;&=\; mcU_1\; e_j i. \end{aligned} \tag{38}$$

But according to (25)

$$e_1 p_1 \;+\; e_2 p_2 \;+\; e_1 e_2 p_3 \;=\; i\;|\vec{p}|\,. \tag{39}$$

which gives

$$\begin{aligned} U_1 \left(\frac{E}{c} + |\vec{p}| \right) i &= mc U_2 i \\ U_2 e_j \left(\frac{E}{c} - |\vec{p}| \right) i &= mc U_1 e_j i \end{aligned} \tag{40}$$

or

$$\begin{aligned} U_1 \left(\frac{E}{c} + |\vec{p}| \right) &= mc U_2 \\ U_2 e_j \left(\frac{E}{c} - |\vec{p}| \right) &= mc U_1 e_j \end{aligned} \tag{41}$$

and

$$\begin{aligned} U_1 \left(\frac{E}{c} + |\vec{p}| \right) &= mc U_2 \\ U_2 \left(\frac{E}{c} - |\vec{p}| \right) e_j &= mc U_1 e_j . \end{aligned} \tag{42}$$

Thus, we finally obtain

$$\begin{aligned} \left(\frac{E}{c} + |\vec{p}| \right) U_1 - mc U_2 &= 0 \\ -mc\, U_1 + \left(\frac{E}{c} - |\vec{p}| \right) U_2 &= 0 \end{aligned} \tag{43}$$

which justifies the above-made assumption concerning the reality of U_1 and U_2. The existence of non-trivial solutions is ensured by

$$\frac{E^2}{c^2} - |\vec{p}|^2 - m^2 c^2 = 0$$

and

$$U_1 = \frac{mc^2}{E + c\, |\vec{p}|} U_2 \tag{44}$$

Thus, we have obtained a triplet of solutions, each one associated with the same mass, but with

$$\begin{aligned} &[\psi_j,\ \psi_k] \neq 0 \\ &[\phi_j,\ \phi_k] \neq 0 \qquad j,k = 1,2,3 \\ &[\psi_j,\ \phi_k] \neq 0 \qquad j \neq k \end{aligned} \tag{45}$$

In addition, in the capacity of $e_j,\ j\ =\ 1,2,3$, one may choose not only the quaternion basis itself but other sets, for example,

$$ie_1i, \quad ie_2i, \quad ie_2e_1i \tag{46}$$

or

$$e_1ie_1, \quad e_2ie_2, \quad e_1e_2ie_1e_2. \tag{47}$$

Notice, however, that (47) do not form a quaternion but

$$e_1ie_1\ + e_2ie_2\ +\ e_1e_2ie_1e_2\ =\ i. \tag{48}$$

Additional knowledge is required in order to define which set is relevant and how it may be associated with the correspondent physical objects.

So far, our discussion has been restricted to the four-dimensional real quadratic division algebra of quaternions. However, it is clear that the equations (24) and (35) serve in an uniform manner also the octonionic extension of the complex Hilbert space .

If the underlying algeraic foundation of the theory is extended to include

the eight-dimensional real quadratic division algebra of octonions, then the corresponding additional set of solutions for the equations (35) may be obtained.

They may have, e.g., the following form:

$$\begin{aligned} \psi &= U_1\ i\ \exp\frac{-i\ (Et\ -\vec{p}\vec{x})}{\hbar} \\ \phi &= U_2\ \exp\frac{-i\ (Et\ -\vec{p}\vec{x})}{\hbar} \end{aligned}$$

(i is given by (25))

$$\left.\begin{aligned} \psi_k &= U_1e_k\exp\frac{-e_k\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\ \phi_k &= U_2\exp\frac{-e_k\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\ \psi_{j_k} &= U_1j_k\exp\frac{-j_k\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \\ \phi_{j_k} &= U_2\exp\frac{-j_k\ (Et\ -\ \vec{p}\vec{x})}{\hbar} \end{aligned}\right. \qquad k\ =\ 4,5,6,7 \tag{49}$$

where $U_1,\ U_2$

$$U_1 \;=\; \frac{mc^2}{E \;+\; c\,|\vec{p}|}\,U_2$$

are real numbers;

$$j_k \;=\; e_k i\,, \qquad k \;=\; 4,5,6,7 \tag{50}$$

and, as before, form a quaternion. This quaternion turns out to form an algebraic foundation of the momentum space and, therefore, the algebraic symmetry between coordinate and momentum spaces may be broken in this formulation.

The particularly symmetric case occurs, if $k \;=\; 7$. Then

$$\begin{aligned} i \;&=\; \frac{e_1 p_1 \;+\; e_2 p_2 \;+\; e_3 p_3}{|\vec{p}|} \\ j \;&=\; \frac{e_4 p_1 \;+\; e_5 p_2 \;+\; e_6 p_3}{|\vec{p}|}. \end{aligned} \tag{51}$$

Indeed, in each case the set of equations (24) and (35) should be supplemented by corresponding leptonic equations, for example, for (51)

$$\begin{aligned} \frac{1}{c}\,\frac{\partial \chi}{\partial t}\,j \;-\; \frac{\partial \chi}{\partial x}e_4 \;-\; \frac{\partial \chi}{\partial y}e_5 \;-\; \frac{\partial \chi}{\partial z}e_6 \;&=\; \frac{mc}{\hbar}\xi \\ -\frac{1}{c}\,\frac{\partial \xi}{\partial t}\,j \;-\; \frac{\partial \xi}{\partial x}e_4 \;-\; \frac{\partial \xi}{\partial y}e_5 \;-\; \frac{\partial \xi}{\partial z}e_6 \;&=\; -\frac{mc}{\hbar}\chi. \end{aligned} \tag{52}$$

Based on the results of M. Zorn [17] that each automorphism of the octonion algebra is completely defined by the images of three "independent" basis units [18], it was demonstrated by M. Günaydin and F. Gürsey [19] that under given automorphism σ we have three quaternionic planes in the space formed by octonion algebra (space of quantum mechanical phases), which undergo rotations by the angles $\phi_1,\ \phi_2,\ \phi_3$, respectively, such that

$$\phi_1 \;+\; \phi_2 \;+\; \phi_3 \;=\; 0 \mod 2\pi \tag{53}$$

remains invariant. The planes $(e_i,\ e_j)$ are determined by the conditions $e_i e_j \;=\; e_k$ and e_k is the fixed point common to all of those planes (compare with (49) and (50)). These results might help to extract the set of independent solutions and to obtain its correct classification.

It is worth mentioning that an alternative arrangement can be also possible. We may consider a septet of solutions, each one associated with the same mass, namely

$$\begin{aligned}\psi &= U_1 e_k \exp \frac{-e_k \ (Et - \vec{p}\vec{x})}{\hbar} \\ \phi &= U_2 \exp \frac{-e_k \ (Et - \vec{p}\vec{x})}{\hbar}\end{aligned} \qquad k = 1, ..., 7 \tag{54}$$

and, perhaps, an additional one in the form

$$\begin{aligned}\psi &= U_1 e_j \exp \frac{-e_j \ (Et - \vec{p}\vec{x})}{\hbar} \\ \phi &= U_2 \exp \frac{-e_j \ (Et - \vec{p}\vec{x})}{\hbar}\end{aligned} \qquad j = 1, ..., 7 \tag{55}$$

where

$$\begin{aligned} j_k &= e_k i e_k \\ j_{k+3} &= e_{k+3} \ j e_{k+3} \\ j_7 &= e_7 \end{aligned} \qquad k = 1, 2, 3 \tag{56}$$

or

$$\begin{aligned} j_k &= i e_k i \\ j_{k+3} &= j e_{k+3} \ j \\ j_7 &= e_7 \end{aligned} \qquad k = 1, 2, 3 \tag{57}$$

with the common fixed point e_7: i and j are given by (51).

The number of independent solutions, which are arranged in such a way, is sharply reduced and serves as a slight reminder of a similar possibility discussed in the literature [20].

Now it may be clarified why we have chosen to discuss the Dirac equations in the Weyl representation. The reason is merely technical. In order to perform clean octonionic calculations, we assume that the solutions of the equations (35) have the form (36), where U_1 and U_2 are real numbers. Then the obtained relations (43) justify the assumption. Notice that the electroweak unification scheme [15] is based on the use of this representation of the Dirac equations; in addition, in that case the solutions behave naturally with respect to the Lorentz transformations.

Let us demonstrate that the Dirac equations in the form (24), as well as the set (35), permit a consistent probabilistic interpretation (here the discussion is restricted to the case where the underlying algebraic basis are quaternions).

Consider formally

$$\begin{aligned}
\frac{1}{c}\ \frac{\partial\psi}{\partial t}\ i\ -\ \frac{\partial\psi}{\partial x_j}e_j\ &=\ \frac{m_1c}{\hbar}\phi\ +\ \frac{m_2c}{\hbar}\phi i\\
-\frac{1}{c}\ \tfrac{\partial\phi}{\partial t}\ i\ -\ \frac{\partial\phi}{\partial x_j}e_j\ &=\ -\frac{m_1c}{\hbar}\psi\ +\ \frac{m_2c}{\hbar}\psi i.
\end{aligned}\qquad j=1,2,3 \tag{58}$$

Then

$$\begin{aligned}
\frac{1}{c}\ \frac{\partial\psi}{\partial t}\ +\ \frac{\partial\psi}{\partial x_j}e_j i\ &=\ -\ \frac{m_1c}{\hbar}\phi\, i\ +\ \frac{m_2c}{\hbar}\phi\\
\frac{1}{c}\ \frac{\partial\phi}{\partial t}\ -\ \frac{\partial\phi}{\partial x_j}e_j i\ &=\ -\frac{m_1c}{\hbar}\psi i\ -\ \frac{m_2c}{\hbar}\psi
\end{aligned}\qquad j=1,2,3 \tag{59}$$

and

$$\begin{aligned}
\frac{1}{c}\ \frac{\partial\bar{\psi}}{\partial t}\ +\ ie_j\frac{\partial\bar{\psi}}{\partial x_j}\ &=\ \frac{m_1c}{\hbar}i\bar{\phi}\ +\ \frac{m_2c}{\hbar}\bar{\phi}\\
\frac{1}{c}\ \frac{\partial\bar{\phi}}{\partial t}\ -\ ie_j\frac{\partial\bar{\phi}}{\partial x_j}\ &=\ \frac{m_1c}{\hbar}i\bar{\psi}\ -\ \frac{m_2c}{\hbar}\bar{\psi}.
\end{aligned}\qquad j=1,2,3 \tag{60}$$

Thus, we have

$$\begin{aligned}
\frac{1}{c}\ \frac{\partial\psi}{\partial t}\bar{\psi}\ +\ \frac{\partial\psi}{\partial x_j}e_j i\bar{\psi}\ &=\ -\frac{m_1c}{\hbar}\phi\, i\bar{\psi}\ +\ \frac{m_2c}{\hbar}\phi\bar{\psi}\\
\frac{1}{c}\ \psi\frac{\partial\bar{\psi}}{\partial t}\ +\ \psi ie_j\ \frac{\partial\bar{\psi}}{\partial x_j}\ &=\ \frac{m_1c}{\hbar}\psi i\bar{\phi}\ +\ \frac{m_2c}{\hbar}\psi\bar{\phi}\\
\frac{1}{c}\ \frac{\partial\phi}{\partial t}\bar{\phi}\ -\ \frac{\partial\phi}{\partial x_j}e_j i\bar{\phi}\ &=\ -\frac{m_1c}{\hbar}\psi\, i\bar{\phi}\ -\ \frac{m_2c}{\hbar}\psi\bar{\phi}\\
\frac{1}{c}\ \phi\frac{\partial\bar{\phi}}{\partial t}\ -\ \phi ie_j\ \frac{\partial\bar{\phi}}{\partial x_j}\ &=\ \frac{m_1c}{\hbar}\phi i\bar{\psi}\ -\ \frac{m_2c}{\hbar}\phi\bar{\psi}.
\end{aligned}\qquad j=1,2,3 \tag{61}$$

Adding all equations (61), we find

$$\frac{1}{c}\left\{\frac{\partial\psi}{\partial t}\bar{\psi} + \psi\frac{\partial\bar{\psi}}{\partial t} + \frac{\partial\phi}{\partial t}\bar{\phi} + \phi\frac{\partial\bar{\phi}}{\partial t}\right\}$$

$$+\left\{\frac{\partial\psi}{\partial x_j}e_j i\bar{\psi} + \psi i e_j\frac{\partial\bar{\psi}}{\partial x_j} - \frac{\partial\phi}{\partial x_j}e_j i\bar{\phi} - \phi i e_j\frac{\partial\bar{\phi}}{\partial x_j}\right\} = 0 \qquad j=1,2,3 \tag{62}$$

Notice that the mass terms vanish separately and, hence, the derivation holds separately for (24) and (35) and from now on it is understood that we consider the solutions of these equations separately.

The equation (62) is invariant under the gauge transformation (15) with i given by (25).

Then we have

$$\frac{1}{c}\frac{\partial}{\partial t}\;\psi\bar{\psi} + \frac{1}{c}\frac{\partial}{\partial t}\;\phi\bar{\phi}$$

$$+\left\{\frac{\partial\psi}{\partial x_j}ie_j\bar{\psi} + \psi e_j i\frac{\partial\bar{\psi}}{\partial x_j} - \frac{\partial\phi}{\partial x_j}ie_j\bar{\phi} - \phi e_j i\frac{\partial\bar{\phi}}{\partial x_j}\right\} = 0. \qquad j=1,2,3 \tag{63}$$

Adding equations (62) and (63), we obtain

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\psi\bar{\psi} + \phi\bar{\phi}\right)$$

$$+\frac{1}{2}\{\frac{\partial\psi}{\partial x_j}(e_j i + ie_j)\;\bar{\psi} + \psi(e_j i + ie_j)\;\frac{\partial\bar{\psi}}{\partial x_j} \qquad j=1,2,3 \tag{64}$$

$$-\frac{\partial\phi}{\partial x_j}\;(e_j i + ie_j)\;\bar{\phi} - \phi\;(e_j i + ie_j)\;\frac{\partial\bar{\phi}}{\partial x_j}\} = 0$$

or

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0 \tag{65}$$

where

$$\rho = \psi\bar{\psi} + \phi\bar{\phi}$$

$$j_k = \frac{c}{2}\left\{\psi(e_k i + ie_k)\;\bar{\psi} - \phi\;(e_k i + ie_k)\;\bar{\phi}\right\}. \qquad k=1,2,3 \tag{66}$$

If i is given by (25), then

$$e_k i \ + \ i e_k \ = \ -\frac{2p_k}{|\vec{p}|}, \quad k = 1, 2, 3. \tag{67}$$

Thus, in accordance with the Galileo, Maxwell and Schrödinger theories, the probability current $\vec{j}$ is proportional to the velocity operator, which is a constant of motion for free particles.

Zitterbewegung [3] phenomenon is absent in this formulation.

3 Conclusion

In account of the experimental information that became available during the last century, it is desirable for the equations of motion for the fundamental fermions to have the following properties:

- the equations should possess $SU(2) \otimes U(1)$ local gauge invariance intrinsically since the electron is not a source of pure electromagnetic radiation but also has the ability to participate in weak interactions;
- the electron is the only particular member of the entire family of fundamental fermions, it is desirable that all fermions are described in the uniform manner;
- there exist three replication of the families of the fundamental fermions;
- leptons do not have a color;
- quarks do have a color;
- each quark appear in triplet associated with the same mass;
- a color is associated with the internal degree of freedom;
- a color symmetry can't be broken.

In contrast to the existing quaternionic formulations [9, 10] of the Dirac equation, we suggested here closely related but essentially different sets of equations that allow description of the free motion for electron and neutrinos, as well as triplets of quarks. The suggested solution possesses $SU(2) \otimes U(1)$ local gauge invariance intrinsically. All obtained solutions have the structure (A1) and (A2). These solutions are substantially different from the standard ones and may yield, therefore, different values for observable physical quantities. We propose to compare and verify them against the existing experimental information. The suggested reexamination may help to decide what is a relevant mathematical framework suitable to achieve a solution of the unification problem for the fundamental interactions.

During the last decades, an enormous progress in the understanding of quantum theory of fields took place. It became almost apparent that we have to deal with four essentially fundamental interactions, which have a similar origin, namely, presence of phases in the quantum mechanical description of the fundamental sources of these fields [21]. In addition, the investigation of the properties of the fundamental sources (leptons, quarks) clearly established that the quantum numbers in addition to electric charge (week hypercharge and color) appear in amazing correspondence with the complex variety of the radiated fields.

The formulation of classical mechanics and the classical theory of fields have demonstrated that the presence of additional interactions requires a suitable generalization of the mathematical language used. Application of the analogy with the structure of classical physics in the framework of functional analysis naturally concentrated around attempts to extent it on all Hurwitz algebras, as the underlying algebraic foundation of the theory.

The above discussion may be considered as an additional step towards realization of a program initiated by E. Schrödinger [22] to treat all of the physics as wave mechanics:

a) The universal mathematical architecture of the physics is given in terms of ten functional - analytical frameworks, suitable to incorporate the results of physical measurements.

Real, complex, quaternion and octonion states with real scalar product should be equivalent to the theory of classical fields. Unification of electromagnetism with gravitation should occur already in the classical field theory.

Complex, quaternion and octonion states with complex scalar product should allow realization of present unification schemes. Notice that pure relativistic quantum electrodynamics does not exist because there are no elementary sources of pure electromagnetic radiation. Neutrino is an elementary source of pure weak radiation.

Quaternion and octonion states with quaternion scalar product should describe wave mechanics of space-time continuum.

Octonion states with octonion scalar product should allow ultimate realization of idea of elementary particles picture of natural phenomena.

b) One expects that the quantum mechanical space-time continuum should be different from its classical counterpart. Perhaps, the spin is not a dynamical variable, but the feature of the quantum mechanical world, namely, the world point is described by the following expression (before inclusion of quantum gravity):

$$X = \begin{pmatrix} t & -e_1x - e_2y - e_3z \\ e_1x + e_2y + e_3z & t \end{pmatrix}$$

It is interesting to construct the correspondent metric space and the application of Least Action should lead to the equations of motion for the fundamental fermions.

c) In the present discussion we consider masses as external phenomenological parameters. However, the structure of octonion quantum mechanics with complex scalar product suggest natural mechanism to generate masses of the fundamental fermions as energy gaps obtained after splitting states of initially degenerated two-level physical system.

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Appendix

This paper is concerned with the relativistic dynamics of single particle states and for this reason we have used only the following results of the particular realization of this program for the quaternionic and octonionic Hilbert spaces with complex scalar products:

1) In the quaternionic extension, quantum mechanical states are represented by

$$\Psi^{(1)}_{\not\subset} = \frac{1}{\sqrt{2}}\binom{f}{fe_1} \quad \text{or} \quad \Psi^{(2)}_{\not\subset} = \frac{1}{\sqrt{2}}\binom{f}{-fe_1} \tag{A1}$$

where $f = f_0 + \sum\limits_{i=1}^{3} f_i e_i$; f_0, f_i are real functions of the space-time coordinates and e_i, $i = 1, 2, 3$ form a basis for the real quadratic division algebra of quaternions;

In the octonionic extension quantum mechanical states are represented by:

$$\Psi^{(3)}_{\not\subset} = \frac{1}{\sqrt{2}}\binom{f}{fe_7} \quad \text{or} \quad \Psi^{(4)}_{\not\subset} = \frac{1}{\sqrt{2}}\binom{f}{-fe_7} \tag{A2}$$

where $f = f_0 + \sum_{i=1}^{7} f_i e_i$; f_0, f_i are real functions of the space-time coordinates and e_i , $i = 1, ..., 7$ are a basis for the real quadratic division algebra of octonions.

The e_1 and e_7 in the definition of the states (A1) and (A2) play the role of a label for the generator of a complex field in the space of one-body states. For example, any one of the quaternionic units or some linear combination of them

$$i = \frac{ae_1 + be_2 + ce_3}{\sqrt{a^2 + b^2 + c^2}} \tag{A3}$$

(a, b, c are arbitrary real numbers) may be used for this purpose. Thus, a definition of this combination cannot be obtained kinematically and turns out to be a matter of the dynamics of single particle states.

2) Consider the general form of operators, induced by the structure (A1) of the vector space.

For the complex linear operators

$$A_z = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{A4}$$

where matrix elements a_{ij} are real operators over quaternions and, in turn, are assumed to be at least z-linear operators, we have

$$\Psi_{\not\subset}^{(1)\,\prime} = \frac{1}{\sqrt{2}} \begin{pmatrix} f' \\ f'e_1 \end{pmatrix} = A_z \Psi_{\not\subset}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f \\ fe_1 \end{pmatrix} \tag{A5}$$

and

$$a_{21}f + a_{22}f_1e_1 = f'e_1 = (a_{11}f + a_{12}fe_1)\, e_1 = a_{11}fe_1 - a_{12}f. \tag{A6}$$

Therefore,

$$a_{12} = -a_{21}; \quad a_{11} = a_{22}. \tag{A7}$$

The restrictions (A7) on the matrix elements of the operator (A4), obtained for the states of the form $\Psi_{\not\subset}^{(1)}$, are also valid if one considers the transformations

$$\Psi_{\not\subset}^{(2)\,\prime} = A_z\, \Psi_{\not\subset}^{(2)}. \tag{A8}$$

However, for the operators transforming the state $\Psi_{\not\subset}^{(1)}$ into the state $\Psi_{\not\subset}^{(2)}$ (and vice versa),

$$\Psi_{\not\subset}^{(2)\,\prime} = \frac{1}{\sqrt{2}} \begin{pmatrix} f' \\ -f'e_1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} f \\ fe_1 \end{pmatrix}, \tag{A9}$$

we have

$$a_{21} = a_{12}; \quad a_{11} = -a_{22}. \tag{A10}$$

Thus, we have obtained two possible types of complex linear operators, either

$$A_z^{(1)} = \begin{pmatrix} a_{11} & a_{12} \\ -a_{12} & a_{11} \end{pmatrix} \tag{A11}$$

or

$$A_z^{(2)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix} \tag{A12}$$

We remark that the matrix elements a_{11} and a_{12} do not commute.

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