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Thermal quantum Hall effect in disordered superconductors *

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Abstract

We study the symmetry class for localization which arises from models of noninteracting quasiparticles in disordered superconductors that have neither time-reversal nor spin-rotation invariance. Two-dimensional systems in this category, which is known as class D, can display phases with three different types of quasiparticle dynamics: metallic, localized, or with a quantized (thermal) Hall conductance. Correspondingly, they can show a variety of delocalization transitions. We illustrate this behavior by investigating numerically the phase diagrams of network models with the appropriate symmetry and show the appearance of the metallic phase. We also study level statistics for this symmetry class and find that the nearest neighbor spacing distribution (NNSD) at the critical energy follows the Wigner's surmise for Gaussian Unitary Ensembles (GUE), reflecting therefore only "basic" discrete symmetries of the system (time reversal violation) and ignoring particle-hole symmetries and other finer details (criticality). In the localized regime level repulsion is suppressed.

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1 One-channel network model: introduction

The original network model [1] was proposed to describe transitions between plateaux in the quantum Hall effect (QHE). QHE is realized in a two-dimensional electron gas subjected to a strong perpendicular magnetic field and a random potential. When random potential varies smoothly (its correlation length is much larger than the magnetic length), a semiclasscial description becomes relevant: electrons move along the lines of constant potential. When two equipotential lines come close to each other (near a saddle point) tunneling is feasible. In the network model, electrons move along unidirectional links forming closed loops in analogy with semiclassical motion on contours of constant potential. Scattering between links is allowed at nodes in order to map tunneling through saddle point potentials. Propagation along links yields a random phase ϕ, thus links are presented by diagonal matrices with elements in the form $\exp(i\phi)$. Transfer matrix for one node relates a pair of incoming and outgoing amplitudes on the left to a corresponding pair on the right; it has the form

$$\mathbf{T} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{1}$$

The node parameter θ is related to the electron energy in the following way

$$\epsilon = -\frac{2}{\pi}\ln(\sinh\theta), \tag{2}$$

where ϵ is a relative distance between the electron energy and the barrier height. It is easy to see that the most "quantum" case (equal probabilities of scattering to the left and to the right) is at $\epsilon = 0$ ($\theta = 0.8814$), in fact numerical calculations show that there is an extended state at that energy.

Numerical simulations on the network model are performed in the following way: one studies system with fixed width M and periodic boundary conditions in the transverse direction. Multiplying transfer matrices for N slices and then diagonalizing the resulting total transfer matrix , it is possible to extract the smallest Lyapunov exponent λ (the eigenvalues of the transfer matrix are $\exp(\lambda N)$). The localization length ξ_M is proportional to $1/\lambda$. Repeating calculations for different system widths and different energies allow one to show that the localization length ξ_M satisfies the scaling relation

$$\frac{\xi_M}{M} = f\left(\frac{M}{\xi(\epsilon)}\right). \tag{3}$$

In the QHE the thermodynamic localization length $\xi(\epsilon) \sim |\epsilon|^{-\nu}$ and $\nu = 2.5 \pm 0.5$. This is the main result [1] which is in a good agreement with experimental data for spin-split resolved levels [2], numerical simulations using other models [3], and the semiclassical argument [4, 5] that predicts $\nu = 7/3$.

2 New symmetry class D: phase diagram

Altland and Zirnbauer [6] considered properties of quasiparticles in disordered superconductors that are governed by a quadratic Hamiltonian which may include effects of disorder in both the normal part and the superconducting gap function. Such Hamiltonians are representatives of a set of symmetry classes different from the three classes which are familiar both in normal disordered conductors and in the Wigner-Dyson random matrix ensembles. A list of additional random matrix ensembles, determined by these new symmetry classes, has been established. These additional random matrix ensembles describe zero-dimensional problems, and are appropriate to model a small grain of a superconductor in the ergodic highly conducting limit. In our work we have extended the study of class D into two dimensions and found transitions between metallic, localized, or quantized Hall phases for quasiparticles.

This symmetry may be realized in superconductors with broken time-reversal invariance, and either broken spin-rotation invariance (as in d-wave superconductors with spin-orbit scattering) or spinless or spin-polarized fermions (as in certain p-wave states). The associated changes in quasiparticle dynamics must be probed by energy transport, since neither charge density nor spin are conserved. A Bogoliubov-de Gennes Hamiltonian with this symmetry may be written in terms of a Hermitian matrix [6]. The corresponding time evolution operator is real, restricting the generalized phase factors to be $\mathrm{O}(N)$ matrices for a model in which N-component fermions propagate on links, and to the values ± 1 for $N = 1$, the case that was studied. We define two models: uncorrelated O(1) model, where phases on the links are independent random variables, and the model first introduced by Cho and Fisher (CF) [7] where scattering phases with the value π appear in correlated pairs (see details below). Each model has two parameters: the first one is a disorder concentration W, such that there is a probability W $(1 - W)$ to have phase 0 (π) on a given link. The second parameter is an energy ϵ describing scattering at the nodes. We have found [8] that in the uncorrelated O(1) model, all states are extended independent of ϵ and W.

For the CF model, the phase diagram (presented in Fig. 1) in the ϵ-W plane has three distinctive phases: metallic, and two insulating phases characterized by different Hall conductivities. The sensitivity to the disorder is a distinctive feature of class D.

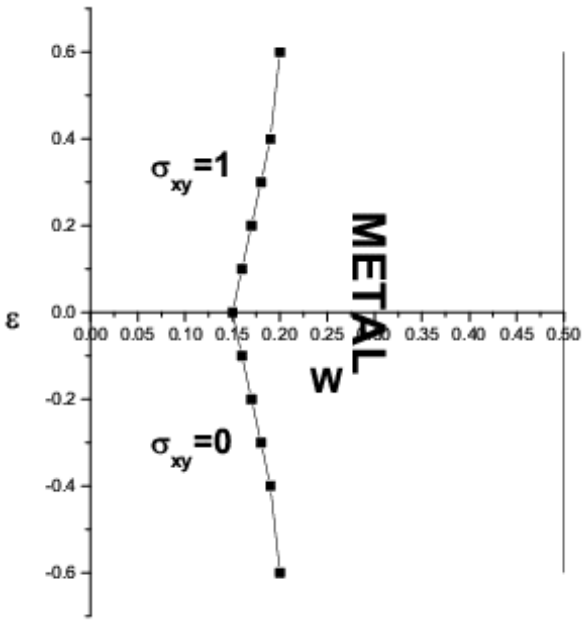

Figure 1: The phase diagram of the CF model obtained from our numerical calculations.

The existence of a region of extended states means that the smallest Lyapunov exponent at each particular energy is zero or extremely small. We wish to discuss this point in detail to demonstrate the power of the CC model. First we present an analytic argument for this result, and then show how to modify the numerical algorithm employing additional symmetries of the system that increases accuracy of calculations.

Consider first the uncorrelated O(1) model with M=2. It has two eigenvectors $(1,-1)^T$ and $(1,1)^T$. The effect of one node and one link transfer matrix on the first eigenvector is

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \exp(-A\theta) \begin{pmatrix} 1 \\ -A \end{pmatrix} \qquad (4)$$

and after many iterations

$$... = \exp[(-A - AB - ABC - ...)\theta] \begin{pmatrix} 1 \\ -ABC... \end{pmatrix} \tag{5}$$

where A, B, C, ... assume values $+1$ with the probability W, and -1 with the probability $1 - W$. The same procedure with the eigenvector $(1, 1)^T$ produces the same result as Eq. (7), with all signs reversed. The weighted averaged value of the exponent (taking into account contributions of both eigenvectors) defines the Lyapunov exponent. We therefore need to find the relative weights of the two eigenvectors. The ergodicity of the system implies that after many iterations, the expression $ABC...$ equals $+1$ (-1) with probability α $(1 - \alpha)$ for some constant α. Assuming the same probability after the next step, we find $\alpha W + (1 - \alpha)(1 - W) = \alpha$ which immediately gives $\alpha = 1/2$, thus both eigenvectors have the same relative weight, and their contributions to the Laypunov exponent cancel each other exactly. We therefore conclude that for M=2, the Lyapunov exponent is exactly zero independent of θ (ϵ) and W.

The decomposition of the transfer matrix for the CF model gives

$$\begin{pmatrix} \cosh A\theta & \sinh A\theta \\ \sinh A\theta & \cosh A\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}. \tag{6}$$

The same exercise for the CF model produces $\exp(A\theta)$ for both eigenvectors, the Lyapunov exponent then is zero only when $< A >= 0$, i.e. for $W = 1/2$.

The standard method for calculating numerically Lyapunov exponents involves application of transfer matrix for successive slices of the system on a set of M orthogonal vectors, and imposing orthogonality by means of the Gram-Schmidt procedure [9]. If all Lyapunov exponents are separated by gaps, this set of vectors converges to the eigenvectors of $T^T T$ associated with the first M exponents (the width of the system is $2M$). Convergence rates are determined by the sizes of gaps between adjacent exponents. In the present case, convergence rates are seriously reduced if the smallest positive Lyapunov exponent ν_1 approaches zero, so that the gap between the smallest exponents vanishes. Moreover, numerical noise ultimately limits the extent of convergence, and leads to an erroneously large value for ν_1. To overcome this flow the following modification of the numerical algorith was proposed [8, 10]. Consider the transfer matrix T of the whole system in more detail. It has the polar decomposition

$$T = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} A_3^T & 0 \\ 0 & A_4^T \end{pmatrix}, \tag{7}$$

where $A_1, ...A_4$ are $M \times M$ real orthogonal matrices and γ is an $M \times M$ real diagonal matrix. It follows that $T^T T$ is diagonalized by the transformation $B^T T^T T B$, where

$$B = \begin{pmatrix} A_3 & A_3 \\ A_4 & -A_4 \end{pmatrix}. \tag{8}$$

We then impose on the M pertinent vectors (beyond simple orthogonality) an additional constraint that their first M components separately form an orthogonal matrix A_3, and their last M components from A_4, as is evident from Eq. (4). This procedure drastically improves convergence and accuracy of the calculations, confirming analytical arguments presented above for both systems.

3 Level statistics: introduction

Statistics of energy levels in disordered systems is an important tool in determining their transport properties as well as their critical behavior. A central quantity in this study is the nearest neighbor spacing distribution (NNSD) denoted by $p(s)$. Here the random variable s is the (fluctuating) level spacing under the proviso that the local average of the density of states is energy independent (otherwise, a proper unfolding procedure is required). The distribution $p(s)$ involves all N point correlation functions of the pertinent Green function and hence, it is generally not available in a closed form.

In dealing with disordered systems, it is useful to distinguish between systems undergoing an Anderson type metal-insulator transition, and those characterized by quantum Hall (QH) like transition where, in the thermodynamic limit, critical state energies are isolated points occuring between continuous intervals of localized states energies. As for level statistics pertaining to disordered systems of the Anderson metal-insulator transition kind, there is a couple of important properties which are well established: 1) Under certain conditions it is expected to be represented (on the metallic side) by random matrix spectra[11]. To be more precise, it is well described by one of the corresponding Gaussian ensembles, GOE (Gaussian orthogonal), GUE (Gaussian unitary), and GSE (Gaussian simplectic), depending on the symmetry class to which the physical system belongs. The main condition is that the corresponding energy intervals are smaller than the Thouless energy. 2) It has been shown[12] that in the limit of an infinite system there are only three types of distributions $p(s)$. They are the Poisson

law for the insulating regime, the Wigner surmise for the metallic domain and a third one for the critical region. On the other hand, for systems in the second group (such as the QHE) there is no similar analysis. The main difficulty is related to the fact that in the absence of a metallic regime, it is not possible to approach the critical point from the metallic regime using the powerful tool of expansion in the small parameter $1/g$ (here g is the dimensionless conductance). Common sense suggests that the distribution follows the Poisson law in the insulating part of the spectra while again, the distribution in the critical region is different, and related to the relevant universality class. For the QH transition this is supported by numerous numerical calculations[13].

We present results for class D of disordered superconductors that have neither time-reversal nor spin-rotation invariance. NNSD at the critical energy (after unfolding) coincides with Wigner's surmise, which was also the result for the original $U(1)$ network model. We find the density of states (DOS) to have a periodic structure (period $\pi/2$) as one would expect from the form of the unitary operator. Beside the critical region, we also present results for the localized regime and show that level repulsion disappears. Thus, although the $U(1)$ and class D models describe different systems and have different phase diagrams, yet the NNSD in the critical region is the same, depending only on the broken time-reversal symmetry. The fact that class D obeys the particle-hole symmetry is not reflected in its NNSD. Our findings are in agreement with recent works [14] where it is argued that k-body embedded Gaussian ensembles of random matrices for sufficiently high rank k of the random interaction behave generically (i.e. in order to have exact RMT results it is not necessary for the Hamiltonian to be a full random matrix).

4 Numerical model and unfolding procedure

If the network forms a torus, then on every link the electron motion appears once as an outgoing one and once as an incoming one. The collection of relations between incoming and outgoing amplitudes defines the system's S matrix or rather, a discrete-time unitary evolution operator, $U(\epsilon)$ [16, 17]. The eigenphases of U serve as input for level statistics analysis. For a square network of $\mathcal{N} \times \mathcal{N}$ nodes , U is an $(2 \times \mathcal{N}^2) \times (2 \times \mathcal{N}^2)$ unitary matrix. The action of U on a vector Ψ of flux amplitudes, defined on the start of each link maps system on itself, providing therefore an *implicit* eigenvalue equation $U(\epsilon)\Psi = \Psi$. Since the dependence of the matrix elements of U on ϵ is

complicated, it is practically impossible to find solutions of that equation (even numerically). Instead, it has been suggested[17] to find the eigenvalues of the equation

$$U\Psi_n = \exp[i\omega_n(\epsilon)]\Psi_n, \tag{9}$$

and to study statistics of ω_n for a given ϵ. The rationale behind it is twofold. First, there are many states even in the narrow window near a paricular energy ϵ to provide good statistics. Second, the behavior of the curves $\omega_n(\epsilon)$ is smooth enough, therefore statistics of ω_n for a given ϵ is expected to be the same as statistics of ϵ_n for $\omega = 0$ (which are the true energy eigenvalues). We argue that the second hypothesis is justified only after a proper unfolding procedure is executed. Taking into account that level statistics should be manifested for each individual sample (as in the study of nuclear spectra), the spectrum of each sample should then be properly unfolded. Here we use the dimensionless unfolded distance between two levels

$$\Delta s_n = 2k\frac{E_{n+1} - E_n}{E_{n+k} - E_{n-k}}, \tag{10}$$

where k is a number of neighbors to be optimized by the requirement of having a constant DOS. This procedure encodes the important local fluctuations of level spacing.

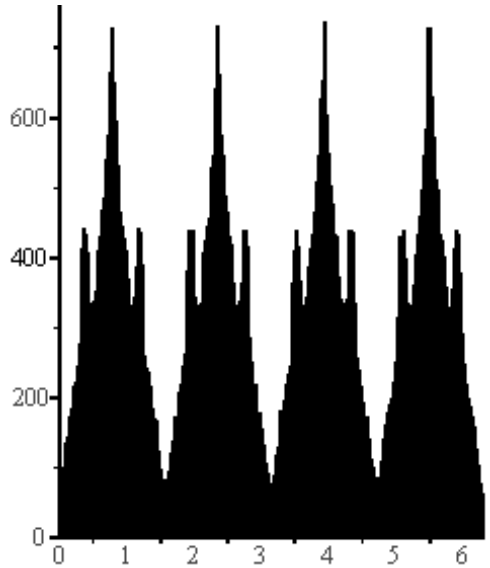

Figure 2: Histogram for the DOS of 50 samples for CF model at the critical point $\epsilon = 0$ and $W = 0.1$.

The phase diagram (Fig. 1) in the (ϵ, W) contains a region of metallic states, and two distinct localized domains, which can be identified as regions with different quantized thermal Hall conductance. There is a critical state at $\epsilon = 0$ for any W.

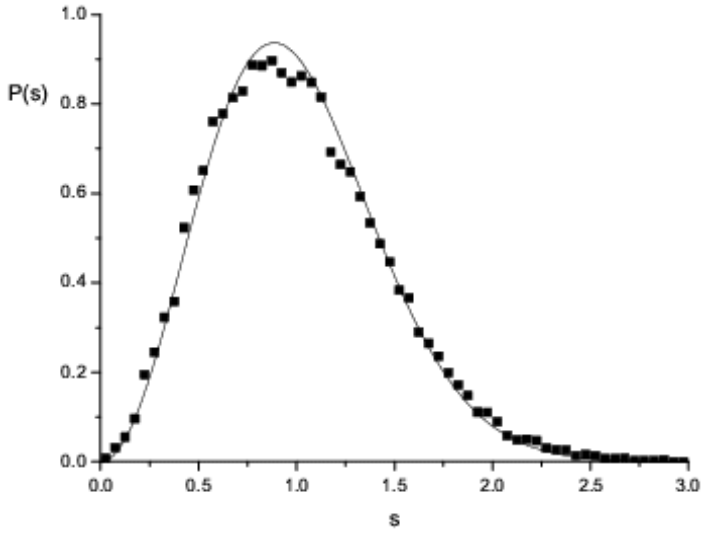

Figure 3: Nearest neighbor spacing distribution $p(s)$ for the CF model at $\epsilon = 0$ and $W = 0.1$ (metallic regime). The solid line is Wigner's surmise for GUE.

In our numerical simulations we have studied 50 different network systems, of size $(2 \times 32^2) \times (2 \times 32^2)$ on the critical line $\epsilon = 0$ and disorder strength parameter $W = 0.1$. The raw DOS appears to be a periodic function of ω with period $\pi/2$ reflecting the cubic symmetry of the CF model (Fig. 2). The NNSD is presented on the Fig. 3, and are compared with the Wigner surmise for GUE. The agreement is rather evident, and it could not be achieved without unfolding. We are thus convinced that the critical form of the NNSD at the critical line of the CF model (in fact of class D in general) coicides with the GUE. Next, we move away from the critical line and put $\epsilon = 1$, keeping the same value $W = 0.1$ which, according to our phase diagram is well within a localized domain. The results are shown in Fig. 4. They are fitted by the Berry-Robnick approximation[18] , which is an interpolation formula between Wigner's surmise and Poissonian statistics. Usually, the large s behavior is more sensitive to localization than

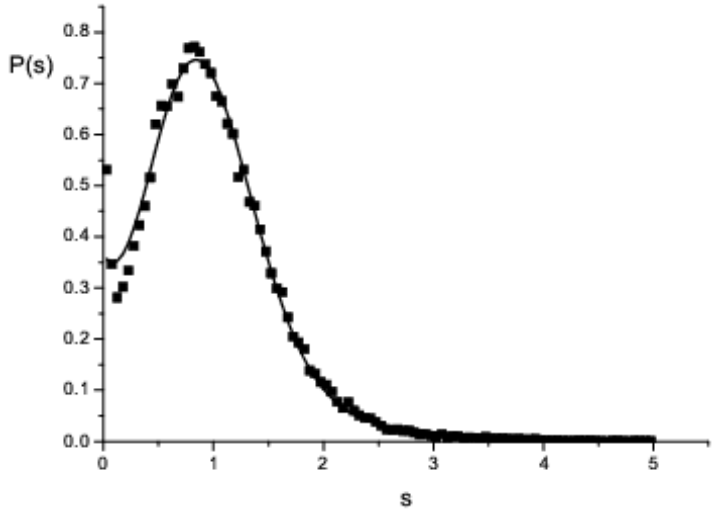

Figure 4: Nearest neighbor spacing distribution $p(s)$ for the CF model at $\epsilon = 1.$ and $W = 0.1$ (localized regime). The solid line is Berry-Robnick fit for a transition from GUE to Poissonian statistics.

the small s one. In other words, even deeply inside the localized regime one still finds level repulsion $p(0) = 0$. Remarkably, for the CF model we find $p(0.025) \approx 0.53$ which cannot be just attributed to statistical error taking into account that we study almost 10^6 energy levels. We have also calculated the compressibility of the spectrum η and have found extremely small value ≈ 0.01 which is in agreement with a classical result for GUE where $\eta \to 0$ for large system sizes.

5 Conclusion

We summarize by mentioning that we have presented results for the class D of disordered superconductors using various applications of the CC model. The model allows some powerful analytical arguments as well as numerical simulations of different physical quantities. The phase diagrams for different realizations of disorder show that while for the uncorrelated $O(1)$ model all states are extended, the CF model has three distinctive phases: metallic, and two insulating phases characterized by different Hall conductivities. The sensitivity to the disorder is a distinctive feature of class D. As for the level statistics, although mappng of a physical problem on a network model

results in correlated and sparse matrices of unitary evolution operators, the results for NNSD seem to agree with the predictions of RMT which assume non-sparse matrices with uncorrelated matrix elements[14]. In the cases studied here this agreement is achieved after a proper local unfolding of the spectra is executed. The main physical result is the following: Despite the occurrence of ten different random matrix symmetry classes according to time-reversal, spin-rotation and particle-hole symmetries, with many different physical properties, some basic characteristics remain intact, depending only on time reversibility and spin rotation invariance. There have been numerous attempts to check whether the form of $p(s)$ in QH like systems deviates from that of GUE[19]. Our results indicate that as far as the network model realization is concerned, $p(s)$ is satisfactorily accounted by the Wigner surmise for the unitary ensemble. The violation of time reversal invariance either by a magnetic field (in QH systems) or spontaneously in unconventional superconductors is the dominant factor, which masks finer details as quantum criticality.

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References

[1] J.T. Chalker and P.D. Coddington, J. Phys. C **21**, 2665 (1988).

[2] S. Koch, R. J. Haug, K. v. Klitzing and K. Ploog, Phys. Rev. Lett. **67**, 883 (1991).

[3] B. Huckestein and B. Kramer, Phys. Rev. Lett. **64**, 1437 (1990).

[4] G.V. Milnikov and I.M. Sokolov, JETP Lett. **48**, 536 (1988).

[5] V. Kagalovsky, B. Horovitz, and Y. Avishai, Europhys. Lett. **31**, 425 (1995).

[6] A. Altland and M.R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).

[7] S. Cho and M.P.A. Fisher, Phys. Rev. B **55**, 1025 (1997).

[8] J.T. Chalker, N. Read, V. Kagalovsky, B. Horovitz, Y. Avishai, and A.W.W. Ludwig, Phys. Rev. B **65**, 012506 (2002).

[9] A. MacKinnon and B. Kramer, Phys. Rev. Lett. **47**, 1546 (1981); Z. Phys. B **53**, 1 (1983); J.L. Pichard and G. Sarma, J. Phys. C **17**, 411 (1981).

[10] F. Merz and J.T. Chalker, cond-mat 0106023 (2001).

[11] B.L. Altshuler and B.I. Shklovskii, Zh. Eksp. Teor. Fiz. **9**, 220 (1986) [Sov. Phys. - JETP **64**, 127 (1986)].

[12] B.I. Shklovskii, B. Shapiro, B.R. Sears, P.Lambrianides, and H.B. Shore, Phys. Rev. B **47**, 11487 (1993).

[13] B. Huckestein, Rev. Mod. Phys. **67**, 357 (1995).

[14] L. Benet, T. Rupp, H.A. Weidenmüller, Phys. Rev. Lett. **87**, 010601 (2001); L. Benet, T. Rupp, H.A. Weidenmüller, Annals Phys. **292**, 67 (2001).

[15] J.T. Chalker and P.D. Coddington, J. Phys. C **21**, 2665 (1988).

[16] C.-M. Ho and J.T. Chalker, Phys. Rev. B **54**, 8708 (1996).

[17] R. Klesse and M. Metzler, Europhys. Lett. **32**, 229 (1995).

[18] M. Robnik, J. Phys. A **20**, 495 (1987).

[19] M. Feingold, Y. Avishai and R. Berkovits, Phys. Rev. B **52**, 8400 (1995); V.E. Kravtsov, I.V. Lerner, B.L. Altshuler, and A.G. Aronov, Phys. Rev. Lett. **72**, 888 (1994).