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Fractional quantum Hall effect and vortex lattices *

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Abstract

It is demonstrated that all observed fractions at moderate Landau level fillings in the quantum Hall effect can be obtained without recourse to the phenomenological concept of composite fermions. The possibility of having special topologically nontrivial many-electron wave functions is considered. Their group classification indicates special values of the electron density in the ground states separated by a gap from excited states.

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The experimental discovery of Integer Quantum Hall Effect (IQHE) by K. Klitzing [1] and Fractional Quantum Hall Effect (FQHE) by Tsui, Stormer and Gossard [2] was one of the most outstanding achievements in condensed matter physics of the last century. Despite the fact that more than twenty years have elapsed since the experimental discovery of Quantum Hall Effect, the theory of this phenomenon is far from being complete (see reviews [3, 4]). This is primarily true for the Fractional Quantum Hall Effect, which necessitates the electron-electron interaction and can by no means be explained by a one-particle theory, in contrast to the IQHE. The most successful variational many-electron wave function for explaining the

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1/3 and other odd inverse fillings was constructed by Laughlin [5, 6]. The explanation of other observed fractions was obtained by various phenomenological hierarchial schemes with construction of the "daughter" states from the basic ones (Haldane [7], Laughlin [8], B. Halperin [9]).

In those works, the approximation of extremely high magnetic field was used, in which one is restricted to the states of the lowest Landau level. However, this does not conform to the experimental situation, where the cyclotron energy has the order of the mean energy of electron–electron interaction. Moreover, this approach encounters difficulties in generalizing to other fractions. Computer simulations give a rather crude approximation for the realistic multiparticle functions, because the number of particles in the corresponding calculations with modern computers does not exceed several tens.

The most successful phenomenological description is given by the Jain's model of "composite" fermions [10, 11], which predicts the majority of observed fractions. According to this model, electrons are dressed with magnetic flux quanta concentrated in an infinitely narrow region around each electron. It is assumed that an even number of flux quanta provides that these particles are fermions. The inclusion of this additional magnetic field in the formalized theory leads to the so-called Chern– Simons Hamiltonian. Below I will describe this approach to [12].

One can perform canonical transfomation of the basis of the many particle wave functions $\Phi \to \exp i\hat{S}\Phi$, where $\hat{S}$ a Hermitian operator:

$$\hat{S} = \int h(\boldsymbol{\xi}, \boldsymbol{\xi}')\psi^+(\boldsymbol{\xi})\psi(\boldsymbol{\xi})\psi^+(\boldsymbol{\xi}')\psi(\boldsymbol{\xi}')d^2\xi d^2\xi'. \tag{1}$$

Here ψ^+, ψ are one particle field operators and $h(\boldsymbol{\xi}, \boldsymbol{\xi}') = h(\boldsymbol{\xi}', \boldsymbol{\xi})$. This transformation gives canonical transformation of the field operators

$$\tilde{\psi} = \exp(-i\hat{S})\psi\exp(i\hat{S}), \quad \tilde{\psi}^+ = \exp(-i\hat{S})\psi^+\exp(i\hat{S}),$$

which have the form

$$\tilde{\psi}(\mathbf{r}) = e^{i\hat{\alpha}(\mathbf{r})}\psi, \quad \tilde{\psi}^+(\mathbf{r}) = \psi^+(\mathbf{r})e^{-i\hat{\alpha}(\mathbf{r})}. \tag{2}$$

Here $\hat{\alpha}(\mathbf{r}) = 2\int h(\mathbf{r}, \boldsymbol{\xi})\psi^+(\boldsymbol{\xi}, \boldsymbol{\xi})d^2\xi$, electron spins are assumed to be identical and spin indices are omitted. "Dressing" electrons with the "flux" is achieved by

$$h(\mathbf{r} - \boldsymbol{\xi}) = K\arctan\frac{r_y - \xi_y}{r_x - \xi_x}, \alpha(\mathbf{r}) = 2K\int \arctan\frac{r_y - \xi_y}{r_x - \xi_x}d^2\xi\psi^+(\boldsymbol{\xi})\psi(\boldsymbol{\xi}). \tag{3}$$

It is assumed that operator $\hat{S}$ is single-valued for any realization of electron coordinates (i.e. the electron density is the sum of δ- functions) which results in integer values of K. Transformed Hamiltonian is obtained by direct substitution of (2) into the hamiltonian of interacting electrons:

$$\tilde{H} = \frac{1}{2m}\int \tilde{\psi}^{+}(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A} + \frac{e}{c}\hat{\mathbf{a}})^2\tilde{\psi}d^2r$$

$$+\frac{1}{2}\int V(\mathbf{r}-\mathbf{r}')\tilde{\psi}^{+}(\mathbf{r})\tilde{\psi}^{+}(\mathbf{r}')\tilde{\psi}(\mathbf{r}'\tilde{\psi}(\mathbf{r})d^2rd^2r'$$

$$\mathrm{curl}\hat{\mathbf{a}} = 2\mathrm{K}\Phi_0\tilde{\psi}^{+}(\mathbf{r})\tilde{\psi}(\mathbf{r}),$$

where Φ_0 is the flux quantum. This Hamiltonian is called Chern-Simons Hamiltonian with an artificial 6-fermion interaction due to the "dressing" with no small parameters. Further treatings of this hamiltonian usually use mean field approximation for the operator of "effective" field $\hat{a}$:curl$\hat{\mathbf{a}}$ = $2\mathrm{K}\Phi_0\mathrm{n_e}$, where n_e is the average electron density. In this approximation we have an "effective" magnetic field additional to the external one. At the integer fillings of Landau levels (LLs) the total field should contain cyclotron gaps. That gives special Jain's fractional fillings of the LLs in the external magnetic field $\nu = q/(1-2Kq)$ where q is an integer. The choice $K = -1$ gives most part of observed fractions. Performing any real calculations is a very difficult task because the mean field approximation is used quite arbitrary and there is no small parameter to consider fluctuation. Performed calculations of the "Fermi liquid" state at $\nu = 1/2$ ($q \to \infty$) give infinite effective mass [12].

However, the theory of FQHE can likely be developed on a different physical basis that is associated with the existence of topological textures stable to finite deformations. The topological classification of multiparticle wave functions is a rather complicated mathematical problem and, to my knowledge, no simple and at the same time effective definition of topological classes is presently available. Classification of topological excitations is well elaborated for a ferromagnetic 2D electron gas in a strong magnetic field with the filling $\nu = 1$ (skyrmions [13, 14]).

As an alternative to the composite fermion approach it is possible to consider a more simple canonical transformation with the same objectives but conserving standard 4-fermionic interaction and connected with the topological textures of the vortex lattice type. I assume a canonical transformation with the operator $\hat{S}$ of the form

$$\hat{S} = \int \psi^{+}_{\mu}(\mathbf{r})V_{\mu\nu}(\mathbf{r})\psi_{\nu}(\mathbf{r})d^2r. \qquad (4)$$

I introduce here some additional index for the electron operators. It can be either spin or some kind of isospin. In fact, a 2D electron system is obtained by filling only one quantum state for the electron motion in transverse direction. In that sense the electron states are ordered in this direction and isospin acquires only one value. The energetical cost of other transverse states with different value of isospin defines the size of an area with the rotated isospin. Therefore, we use some spinors ψ_μ with the exact meaning of spinor index not essential for future. For simplicity I consider ordinary spin indices. The transformed spinors $\chi_\mu = e^{-i\hat{S}}\psi_\mu e^{i\hat{S}}$ have the form $\chi_\mu(\mathbf{r}) = U_{\mu\nu}(\mathbf{r})\psi(\mathbf{r}), \chi^+_\mu(\mathbf{r}) = \psi^+(\mathbf{r})U^*_{\mu\nu}(\mathbf{r})$ of spinors after nonuniform rotation by the unitary matrix $\hat{U} = \exp i\hat{V} = U_z(\alpha(\mathbf{r})U_y(\beta(\mathbf{r})U_z(\alpha(\mathbf{r})$ where $\alpha, \beta, \gamma = \alpha$ are three Euler angles and the lower indices denote the axis of rotation.

After the canonical transformation, the Lagrangian of interacting electrons takes the form (in the system of units where external magnetic field $B = 1, l_B = 1$, and $\hbar = 1$)

$$L = \int \left[i\chi^+ \frac{\partial \chi}{\partial t} - \frac{1}{2m}\chi^+(-i\nabla + A_0^{\vec{}} + \hat{}\,)^2 \Omega \right] d^2r$$
$$+ \frac{1}{2}\int V(\vec{\mathbf{r}} - \vec{\mathbf{r}'})\chi^+(\vec{\mathbf{r}})\chi^+(\vec{\mathbf{r}'})\chi(\vec{\mathbf{r}'})\chi(\vec{\mathbf{r}})d^2r d^2r', \tag{5}$$

where

$$\hat{\Omega} = -iU^+\boldsymbol{\nabla}U = \boldsymbol{\Omega}^l\sigma_l,$$

σ_l are Pauli matrices,

$$\boldsymbol{\Omega}^z = \frac{1}{2}(1 + \cos\beta)\boldsymbol{\nabla}\alpha,$$

$$\boldsymbol{\Omega}^x = \frac{1}{2}(\sin\beta\cos\alpha\boldsymbol{\nabla}\alpha - \sin\alpha\boldsymbol{\nabla}\beta),$$

$$\boldsymbol{\Omega}^y = \frac{1}{2}(\sin\beta\sin\alpha\boldsymbol{\nabla}\alpha + \cos\alpha\boldsymbol{\nabla}\beta),$$

and $V(\mathbf{r} - \mathbf{r}')$ is the Coulomb interaction. It is assumed that $\gamma = \alpha$, because the angle γ plays an auxiliary role, eliminating singularities of the matrix $\hat{U}$. The spinors ψ and ψ^+ are the electron-field operators obeying the Fermi commutation rules. One can readily verify that χ^+ and χ satisfy the same commutation rules. The new Lagrangian is formally equivalent to the initial one with $\hat{\Omega} = 0$. Hence, this Lagrangian gives electronic states corresponding to $\hat{\Omega} = 0$, since one can always perform inverse transformation. However, one may attempt to look for any other states that are characteristic of the

Lagrangian with $\hat{\Omega} \neq 0$. This program can be successfully implemented in the case where U changes only slightly at a distance of the order of magnetic length $l_B = (\hbar c/eB)^{1/2}$ and all $l_B|\mathbf{\Omega}^l|$ are small. At large distances, $\beta \to 0$ so that the matrix U only rotates spinors around the z axis, which aligns with the spin orientation in a homogeneous ferromagnet and endows them with a nontrivial phase. The desired electronic state for operators χ and χ^+ can be obtained perturbatively for small $\hat{\Omega}$ from a uniform ferromagnetic state for operators χ. The existence of a topological number

$$K = \frac{1}{2\pi} \int \mathrm{curl}\mathbf{\Omega}^z \mathrm{d}^2\mathrm{r}$$

which is determined by the number of revolutions through the angle $\alpha(\vec{r})$ upon going around the infinite far contour, is a nontrivial topological requirement. The value of $K \neq 0$ precisely defines the wave-function topological class and makes the wave-function deformation into the trivial ferromagnetic state with identical directions of all ψ spinors impossible. Thus, $\hat{\Omega}$ with different K characterize the topologically different classes of multiparticle wave functions. The condition $\beta = \pi$ at the points of $\alpha(\vec{r})$ singularity (of the polar-angle type) guarantees the absence of singularities for $\hat{\Omega}$. This approach was suggested in [15]; various physical quantities were calculated in [16] in the lowest order of perturbation theory. The results for the syrmion energy coincide with those obtained by other methods (see [12, 13]). The quantity $\mathrm{curl}\mathbf{\Omega}^z$ plays the role of an additional effective magnetic field, and this field is a collective property of the multiparticle wave function rather than an attribute of an individual electron. Calculations of the electron density, energy, and spin density can be, in principle, carried out up to any order in the derivatives of matrix U.

This example demonstrates the method of determining isolated topological excitations. However, this approach can be extended to the analysis of the texture and a multiparticle wave function corresponding to the finite density of topological number K on a 2D plain. The analysis of arbitrary textures of this type for $\hat{\Omega}$ involves great methodological difficulties and, likely, bears no direct relation to the ground-state classification. We, therefore, assume that these textures are near-periodic, so that the average-spin field is periodic. Essentially, I want to construct periodic vortex lattice. Let us consider an elementary cell. We assume that the average spin vector at the elementary cell boundary has a constant value and is aligned with the z axis in the spin space. Thus the angle β is assumed to be a periodic function in 2D plane, with $\beta = 0$ at the elementary cell boundaries. The angle α is assumed to possess vortex singularity at some point inside each unit cell,

for which we assume that $\beta = \pi$ in order to eliminate the singularities of $\hat{\Omega}(\mathbf{r})$. One can set, for example, $\alpha = \sum \alpha_i$, where the summation goes over all elementary cells and $\alpha_i(\mathbf{r})$ is the polar angle centered inside the i elementary cell. The detailed form of $\alpha(\mathbf{r})$ is not very important because $\hat{\Omega}$ form some vector potential and gradient transformation can eliminate part of this matrix. The circulations of $\mathbf{\Omega}^k$ along the sides of elementary cell are easily calculated and give $\oint \mathbf{\Omega}^z d\mathbf{l} = 2\pi K$, $\oint \mathbf{\Omega}^{x,y} d\mathbf{l} = 0$, where K is the winding number for α inside each cell. Thus we have a constant total flux through each elementary cell for positive spin in the direction of external magnetic field $\Phi = Ba_c + \Phi_0 K$ (a_c is the elementary cell area) and periodical "effective" magnetic field due to $\mathrm{curl}\mathbf{\Omega}^{\mathrm{x,y}}$ with zero total flux. The interaction in Lagrangian (5) is translation-invariant.

Therefore, each cell is characterized by the same topological number

$$K = \frac{1}{2\pi} \int \mathrm{curl}\mathbf{\Omega}^{\mathrm{z}} \mathrm{d}^2\mathrm{r}$$

that specifies the integer number of the flux quanta for the additional effective magnetic field with the average value $B_{eff} = K/a_c$ over the sample area. Taking ferromagnetic χ and χ^+ as an approximation, the average spin $\mathbf{n}(\mathbf{r})$ gives the K-fold mapping of any elementary cell onto unit sphere. Although the sum $\alpha = \sum \alpha_i$ over all cells is, formally, a periodic function, it diverges. Since only $\sin\alpha$ and $\cos\alpha$ enter the expression for $\hat{\Omega}$, the modulo 2π convergence is sufficient. I will adopt, without proof, that $\mathbf{\Omega}^{x,y}$ can be regularized in a periodic manner.

It is not my intention to calculate electron energy in such textures. This is a rather complicated problem for elementary cell sizes of the order of magnetic length, for which the gradient expansion in $\hat{\Omega}$ is impossible. My goal is to classify the electronic states with the aim of determining certain special density values that correspond to the ground states separated by a gap from the excited states. The problem of numerical calculation of the gap can be posed after the classification of ground states.

We have, in fact, a system of interacting electrons in a periodic effective magnetic field (the sum of the external magnetic field and a periodic vortex "magnetic field" in elementary cells) with nonzero average. The corresponding transformation group consists of the magnetic translations and is the projective representation of the conventional translation group. According to the well-known analysis (Brown [17], Zak [18] for noninteracting electrons, the band spectrum is regular only for a rational number of flux quanta. An irrational number of quanta or a rational number with large coprime numerator and denominator gives a highly irregular structure with

allowed and forbidden bands dense in a certain energy region. One can assume, in the spirit of the Fermi-liquid theory, that the interaction does not affect these spectral features. Restricting oneself to the rational fluxes, we get

$$Ba_c + K\Phi_0 = \frac{p}{q}\Phi_0, \tag{6}$$

where p, q are integers and obtain

$$a_c = \frac{p - Kq}{q}\frac{\Phi_0}{B}$$

for the elementary cell area. The total number of states per unit area, with one electron per elementary cell, determines the electron density

$$n_e = \frac{B}{\Phi_0}\frac{q}{p - qK}$$

and must correspond to the filled set of bands obtained from S/a_c states in the absence of average magnetic field, though in a periodic potential or periodic magnetic field with zero average with the period specified by the elementary cell. Here S is the sample area. Simple analysis states [19] that this initial band is split into q subbands, each being (odd q) q- fold or (even q) $q/2$-fold degenerate, and with the fraction of the number of states in each subband being (odd q) equal to $1/q^2$ or (even q) to $2/q^2$. However, the total number of states in all subbands is S/a_c. One can assume that, even in the presence of interaction, these states are separated from the higher-energy states by the greatest gap. The structure of inner forbidden bands is irrelevant because all lower-lying states are filled. Note that the evenness of the K number is immaterial, because the Fermi commutation rules for the operators χ and χ^+ are fulfilled automatically and have no relation to the topological number K. The occurrence of any specific numbers of vortex-field flux quanta can be dictated by the ground-state energy. The observed fractions in FQHE correspond to the table

$$K = -2, \;\; p = 1$$

q	1	2	3	-5	-2	-3	-4	4	∞
ν	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{4}{9}$	$\frac{1}{2}$

.

These fractions correspond to the famous Jain's rule. Half-filling of the LL $n_e = B/2\phi_0$ in the external field corresponds to vanishingly small effective magnetic field (zero number of flux quanta per elementary cell).

Other observed fractions correspond to

$$K = -1, \;\; p = 1$$

q	-4	4	2
ν	$\frac{4}{3}$	$\frac{4}{5}$	$\frac{2}{3}$

,

where one has double of the fraction 2/3, and

$$K = -1, \;\; p = 2$$

q	-7	-5	5	2
ν	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{5}{7}$	$\frac{1}{2}$

.

Here one has not observed double of the fraction 1/2 with the gap ($B_{eff} \neq 0$).

Thus, I have reproduced the key statement of the theory of composite fermions (Jain's rule) and obtained the explanation of all observed fractions at moderate LLs filling in a unified frame without any hierarchial schemes. These results are a quite crude and in some points hypothetical. The energy gap, the properties of elementary charge excitations, and the conductivity calculations, as well as the analysis of different K and p, q values, are still open questions, and the approach to these problems is still unclear. The preliminary results were published in [20].

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References

[1] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[2] D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[3] *The Quantum Hall Effect*, Eds.: R. Prange and S.M. Girvin (Springer, NY, 1987).

[4] *New Perspectives in Quantum Hall Effects*, Eds.: S. Das Sarma and A. Pinczuk (Wiley, 1997).

[5] F.D.M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).

[6] R.B. Laughlin, Surf. Sci. **142**, 163 (1984).

[7] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[8] R.B. Laughlin, Phys. Rev. B **22**, 5632 (1981).

[9] R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[10] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989).

[11] J.K. Jain, Phys. Rev. B **41**, 7653 (1990).

[12] B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).

[13] S. Sondhi, A. Kahlrede, S. Kivelson, and E. Rezayi, Phys. Rev. B **47**, 16419 (1993).

[14] K. Moon, N. Mori, Kun Yung, S.M. Girvin, A.H. Macdonald, L. Zheng, D. Ioshioka, and S. Zhang, Phys. Rev. B **51**, 5138 (1995).

[15] S.V. Iordanski, S.G. Plyasunov, and I.V. Fal'ko, Zh. Eksp. Teor. Fiz. **115**, 716 (1999) [JETP **88**, 392 (1999)].

[16] S. Brener, S.V. Iordanski, and A.B. Kashuba, Phys. Rev. B **67**, 125309 (2003).

[17] E. Brown, Phys. Rev. **133**, A1038 (1964).

[18] J. Zak, Phys. Rev. **134**, A1602 (1964).

[19] E.M. Lifshitz and L.P. Pitaevski, *Statistical Physics*, Part 2 (Pergamon, New York, 1980).

[20] S.V. Iordanski, Pis'ma Zh. Eksp. Teor. Fiz. **77**, 292 (2003) [JETP Lett. **77**, 247 (2003)]; arXiv: cond-mat/0303227 v1, 12 Mar 2003.