HAIT Journal of Science and Engineering, Volume 1, Issue 1, pp. 101-129

On exact integrability of replica field theories in 0 dimensions: non-Hermitean disordered Hamiltonians *

Eugene Kanzieper

Department of Sciences,
Holon Academic Institute of Technology, Holon 58102, Israel
e-mail: eugene.kanzieper@weizmann.ac.il
Received 19 December 2003

Abstract

Recently discovered exact integrability of zero-dimensional replica field theories [E. Kanzieper, Phys. Rev. Lett. **89**, 250201 (2002)] is examined in the context of Ginibre Unitary Ensemble of non-Hermitean random matrices (GinUE). In particular, various nonperturbative fermionic replica partition functions for this random matrix model are shown to belong to a positive, *semi-infinite* Toda Lattice Hierarchy which, upon its Painlevé reduction, yields exact expressions for the mean level density and the density-density correlation function in both bulk of the complex spectrum and near its edges. Comparison is made with an approximate treatment of non-Hermitean disordered Hamiltonians based on the 'replica symmetry breaking' ansatz. A difference between our replica approach and a framework exploiting the replica limit of an *infinite* (supersymmetric) Toda Lattice equation is also discussed.

PACS: 02.50.-r, 05.40.-a, 75.10.Nr

---

*Presented at Russian-Israeli Conference *Frontiers in Condensed Matter Physics.* Shoresh, Israel, 19-24 October 2003

1 Introduction

How replicas arise.—In physics of disorder, all observables depend in highly nonlinear fashion on a stochastic Hamiltonian hereby making calculation of their ensemble averages very difficult. To determine the latter in an interactionless system, one has to know spectral statistical properties of a single particle Hamiltonian $\mathcal{H}$ contained in the mean product of resolvents, $G(\varepsilon) = \mathrm{tr}\,(\varepsilon - \mathcal{H})^{-1}$. Each of the resolvents can exactly be represented as a ratio of two integrals running over an auxiliary vector field ψ which may consist of either commuting (bosonic) or anticommuting (fermionic) entries. In the random matrix theory limit, when a system Hamiltonian is modelled by an $N \times N$ random matrix $\mathcal{H}$ of certain symmetries, the resolvent $G(\varepsilon)$ equals

$$G(\varepsilon) \;=\; i \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\; \bar{\psi}_\ell \psi_\ell \, e^{-iS_{\mathcal{H}}[\varepsilon;\, \bar{\psi},\psi]} \left(\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\; e^{-iS_{\mathcal{H}}[\varepsilon;\, \bar{\psi},\psi]} \right)^{-1} \tag{1}$$

where $S_{\mathcal{H}}[\varepsilon;\, \bar{\psi}, \psi] = \bar{\psi}_\ell \,(\varepsilon \delta_{\ell\ell'} - \mathcal{H}_{\ell\ell'})\, \psi_{\ell'}$, ψ is an N–component vector $\psi = (\psi_1, \cdots, \psi_N)$, $\bar{\psi}$ is its proper conjugate, and $\mathrm{Im}\,\varepsilon \neq 0$. Summation over repeated Latin indices is assumed.

Although exact, this representation is a bit too inconvenient for a non-perturbative averaging due to the awkward random denominator. To get rid of it, Edwards and Anderson [1] proposed a replica method based on the identity

$$\ln Z = \lim_{n \to \pm 0} \frac{Z^n - 1}{n}. \tag{2}$$

Upon assigning to Z a meaning of a quantum partition function

$$Z(\varepsilon) = i^N \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\; e^{-iS_{\mathcal{H}}[\varepsilon;\, \bar{\psi},\psi]}, \tag{3}$$

the average resolvent $\langle G(\varepsilon) \rangle$ can be determined through the limiting procedure

$$\langle G(\varepsilon) \rangle = \lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial \varepsilon} \langle Z^n(\varepsilon) \rangle \tag{4}$$

involving the average of the partition function

$$Z^n(\varepsilon) = i^{nN} \int \prod_{\alpha=1}^{|n|} \mathcal{D}\bar{\psi}^{(\alpha)} \mathcal{D}\psi^{(\alpha)}\; e^{-iS_{\mathcal{H}}[\varepsilon;\, \bar{\psi}^{(\alpha)}, \psi^{(\alpha)}]} \tag{5}$$

describing n identical noninteracting copies, or replicas, of the initial disordered system (3). The nature of replicated fields $\psi^{(\alpha)}$ in (5) is determined by the sign of n in (2); they are bosonic for $n < 0$ and are fermionic otherwise. Clearly, the procedure (4) assumes mutual commutativity of the replica limit $n \to 0$, an ensemble averaging $\langle \cdots \rangle$, and the differentiation operation $\partial/\partial\varepsilon$.

Contrary to (1), the representations (4) and (5) contain no random denominator hereby making a nonperturbative in disorder calculation of the resolvent $G(\varepsilon)$ viable. Depending on the origin of auxiliary fields $\psi^{(\alpha)}$ in (5), such a disorder averaging supplemented by identification of physically relevant low lying modes of the theory, would eventually result in effective replica field theory [2] (also called a nonlinear replica σ model) defined on either a noncompact [3, 4] or a compact [5] manifold.

Why the replica limit is problematic.—Seemingly innocent at first glance, the above field theoretic construction appears to be counterintuitive and rising fundamental mathematical questions [6]. Indeed, due to a particular integration measure which makes no sense for n other than integers, the average of (5) cannot directly be used to implement the replica limit (4) determined by the behaviour of $Z_n(\varepsilon) = \langle Z^n(\varepsilon) \rangle$ in a close vicinity of $n = 0$. To circumvent this difficulty (which reflects [7] a true, *continuous geometry* [6] of replica field theories), one may at first evaluate the average replica partition function $Z_n(\varepsilon)$ for all $n \in \mathbb{Z}^+$ (or $\mathbb{Z}^-$), and then analytically continue the result into a vicinity of $n = 0$ in order to make taking the replica limit (4) well defined and safe. This route, however, is full of pitfalls.

In the context of mesoscopic physics, the subtleties involved in carrying out the replica limit were discussed at length in Refs. [8-13] . All these studies have debated *whether or not the nonperturbative* [14] *sector of replica field theories is reliable.* This issue is of conceptual importance yet is not pure academic because the replica field theories are among a very few means available to address problems involving both disorder and interactions, about which the famous Efetov's supersymmetry approach (SUSY) – a prime tool in studying noninteracting disordered systems for the last two decades – has nothing to say [15].

Approximate treatment of replicas.—In the early study [8] by Verbaarschot and Zirnbauer, a nonperturbative sector of nonlinear replica σ models was thoroughly examined in the context of the Random Matrix Theory [16] (RMT). Having mapped the problem of eigenvalue correlations in the Gaussian Unitary Ensemble (GUE) of large random matrices onto both bosonic and fermionic replica field theories, these authors had found that the two formulations of nonlinear replica σ model supplied different results for the density-density correlation function, both apparently differing from the correct one firmly established by other methods [16, 17]. These findings led the authors to conclude that the replica method is ‘mathematically ill founded’. The failure of the replica method to correctly account for all nonperturbative contributions to a physical observable was attributed to a *nonuniqueness* of the analytic continuation of replica partition functions in the replica parameter n away from (either negative or positive) integers. This standpoint, recently reiterated by Zirnbauer [12], has formed a prevailing opinion in the literature that the replica method may at best be considered as a perturbative tool not being able to reproduce truly nonperturbative results accessible by alternative SUSY technique.

The paper that challenged the opinion about inner deficiency of replica field theories and triggered their further reassessment was that of Kamenev and Mézard [9]. Based on ideas of replica symmetry breaking originally devised in the theory of spin glasses [18], these authors came up with a procedure that eventually produced nonperturbative results for the GUE density-density correlation function out of fermionic replicas, albeit in an asymptotic region describing evolution of a quantum system at times not exceeding the Heisenberg time. (Subsequently, this approach was applied to a number of problems such as the energy level statistics in disordered metallic grains beyond [10] the RMT limit, spatial correlations in Calogero-Sutherland models [19, 20], a microscopic spectral density of the Euclidean QCD Dirac operator [21], and energy level fluctuations in Ginibre ensembles of non-Hermitean random matrices [22]).

Briefly summarised (for a detailed exposition the reader is referred to original publications [9, 10] as well as to a critical analysis [12] by Zirnbauer), the framework [9] rests on an *approximate* saddle point evaluation of replica partition functions represented in terms of $|n|$–fold integrals containing a large parameter. In doing so, nontrivial saddle point configurations with so-called broken replica symmetry have to be taken into account in order to reproduce nonperturbative results. While leading to asymptotically correct expressions for spectral fluctuations in the Gaussian ensembles possessing unitary, orthogonal and symplectic Dyson’s symmetries, the procedure[9] cannot be considered as mathematically satisfactory because it involves a nonexisting [9, 12] analytic continuation of replica partition functions to a vicinity of $n = 0$, the domain which is crucially important for implementing the replica limit.

Towards exact integrability of replica σ models.—Analysis of Refs. [8-12] (see also Ref. [13]) hints that *approximate* evaluation of replica partition functions is the key point [23] to blame for inconsistencies encountered in the procedure of analytic continuation away from n integers. In such a situation, leaning towards *exact* calculational schemes in replica field theories is a natural move.

A step in this direction was taken in the recent paper [23], where partition functions for a number of fermionic replica σ models were shown to belong to a *positive, semi-infinite* Toda Lattice Hierarchy extensively studied in the theory of nonlinear integrable systems [24]. In conjunction with the τ-function theory [25 - 28] of the six Painlevé transcendents (which are yet another fundamental object in the theory of integrable hierarchies), this observation led to *exact* evaluation of replica partition functions for a number of random matrix ensembles in terms of Painlevé transcendents. Resulting nonperturbative Painlevé representations (which implicitly encode *all* hierarchical inter-relations between partition functions with various replica indices) were used to build a continuation of Z_n's away from $n \in \mathbb{Z}^+$. While not addressing the important issue of *uniqueness* of such an analytic continuation, the route of Ref. [23] has yielded – for the first time – exact nonperturbative results for random matrix spectral statistics out of fermionic replicas.

More recently, Splittorff and Verbaarschot [29, 30] have suggested that such nonperturbative results could directly be obtained from the replica limit of an *infinite* Toda Lattice equation without Painlevé reduction whatsoever. In fact, the approach [29, 30] rests on the observation that, if properly normalised, the fermionic and the bosonic replica partition functions of a zero-dimensional interactionless system form a *single, infinite* (that is, supersymmetric) Toda Lattice Hierarchy belonging to its either positive (fermionic) or negative (bosonic) branch. While greatly simplifying calculations of spectral correlation functions through a remarkable bosonic-fermionic factorisation [30], the framework developed by Splittorff and Verbaarschot is, to a large extent, supersymmetric in nature as it explicitly injects [31] a missing bosonic (or fermionic) information to otherwise fermionic (or bosonic) like treatment.

In the present paper, a detailed account is offered of a nonperturbative approach [23] to zero dimensional fermionic replica field theories which is based on exact Painlevé representation of replica partition functions with the emphasis strongly placed on technical details. Specifically, we focus on the Ginibre ensemble [32] of complex random matrices with no further symmetries. This particular random matrix model is of special interest in the light of recent findings that associate statistical models of normal random matrices with integrable structures of conformal maps and interface dynamics at both classical [33] and quantum scales [34]. (The reader is referred to Ref. [35] for an introductory exposition of these recent developments and to Ref. [36] for a review of other physical applications and extended bibliography).

The paper is organised as follows. In Section 2, we collect the basic definitions and present the major results regarding the Ginibre Unitary Ensemble of random matrices. In Section 3, a fermionic replica field theory approach to non-Hermitean complex random matrices is outlined and integrability of the field theory is established. The integrability which manifests itself in emergence of a positive, semi-infinite Toda Lattice equation for replica partition functions and also results in exact representations of replica partition functions in terms of Painlevé transcendents, eventually culminates in reproducing exact fluctuation formulas for Ginibre complex random matrices. Finally, in Section 4, we make a comparison of our exact approach with the approximate treatment [22] of non-Hermitean disordered Hamiltonians based on the 'replica symmetry breaking' ansatz; we also comment on differences between our approach to replicas and a framework [29, 30] exploiting a replica limit of the infinite Toda Lattice equation.

2 Ginibre unitary ensemble of non-Hermitean random matrices: Definitions and basic results

Preliminaries.—Statistical ensemble of generic $N \times N$ complex random matrices $\mathcal{H} \in \mathbb{C}^{N\times N}$ whose entries are independently distributed in accordance with the Gaussian law [37]

$$P_N(\mathcal{H}) = \pi^{-N^2} \exp\left(-\mathrm{tr}\, \mathcal{H}\mathcal{H}^\dagger\right) \tag{6}$$

has first been introduced in the pioneering work by Ginibre [32]. Throughout the paper, such random Hamiltonians will be denoted as $\mathcal{H} \in \mathrm{GinUE}_N$.

The joint probability distribution function $P_N(z_1, \cdots, z_N)$ of N eigenvalues of the matrix $\mathcal{H}$ is of particular interest. Having Schur-decomposed the $\mathcal{H}$ as $\mathcal{H} = U^\dagger(\mathcal{Z} + \mathcal{R})\, U$ where U is a unitary matrix which is unique up to the phase of each column, $\mathcal{R}$ is a strictly upper-triangular complex matrix, and $\mathcal{Z}$ is a diagonal matrix $\mathcal{Z} = \mathrm{diag}\,(z_1, \cdots, z_N)$ consisting of N complex eigenvalues $\{z_\ell\} = \{x_\ell + iy_\ell\}$ of the $\mathcal{H}$, Ginibre managed to derive the joint probability distribution function of $\{z_\ell\}$ in the form

$$P_N(z_1, \cdots, z_N) = C(N) \prod_{\ell_1<\ell_2=1}^{N} |z_{\ell_1} - z_{\ell_2}|^2 \prod_{\ell=1}^{N} w^2(z_\ell, \bar{z}_\ell) \tag{7}$$

where the weight function $w^2(z, \bar{z})$ equals $w^2(z, \bar{z}) = \exp(-z\bar{z})$. Given the integration measure $d^2 Z_\ell = dx_\ell dy_\ell$, the inverse normalisation constant in (7) is determined to be $C^{-1}(N) = \pi^N \Gamma(N+1)$.

Of primary interest is the p-point correlation function

$$R_p(z_1, \dots, z_p; N) = \frac{N!}{(N-p)!} \int \prod_{\ell=p+1}^{N} d^2 Z_\ell \, P_N(z_1, \cdots, z_N) \tag{8}$$

describing a probability density to find p complex eigenvalues around each of the points $z_1, \cdots, z_p$ while positions of the remaining levels stay unobserved. Quite often, one is also interested in the thermodynamic limit of the correlation function

$$\rho_p(z_1, \cdots, z_p) = \lim_{N\to\infty} \frac{1}{\delta_N^{2p}} R_p\left(\frac{z_1}{\delta_N}, \cdots, \frac{z_p}{\delta_N}; N\right) \tag{9}$$

that magnifies spectrum resolution on the appropriate energy scale δ_N while letting the matrix size N tend to infinity.

The multi-fold integral in (8) can explicitly be evaluated by adopting the Gaudin-Mehta [16] method of orthogonal polynomials originally introduced in the context of Hermitean random matrix theory. It is a straightforward exercise to demonstrate that $R_p(z_1, \cdots, z_p; N)$ admits the determinant representation

$$R_p(z_1, \cdots, z_p; N) = \det\left[K_N(z_k, \bar{z}_\ell)\right]_{k,\ell=1,\cdots,p} \tag{10}$$

involving the scalar kernel

$$K_N(z, z') = w(z, \bar{z})\, w(z', \bar{z}') \sum_{\ell=0}^{N-1} P_\ell(z) P_\ell(z')$$

expressed in terms of polynomials $P_\ell(z)$ orthonormal in the complex plane $z = x + iy$

$$\int d^2 Z \, w^2(z, \bar{z})\, P_k(z) P_\ell(\bar{z}) = \delta_{k\,\ell} \tag{11}$$

with respect to the measure $w^2(z, \bar{z})\, d^2 Z$.

For instance, it follows from (10) that the density of states and the two-point correlation function equal

$$R_1(z; N) = K_N(z, \bar{z}) \tag{12}$$

and

$$R_2(z_1, z_2; N) = K_N(z_1, \bar{z}_1)K_N(z_2, \bar{z}_2) - |K_N(z_1, \bar{z}_2)|^2 \,, \qquad (13)$$

respectively.

For the Gaussian measure, the orthonormal polynomials $P_\ell(z)$ are just monomials

$$P_\ell(z) = \frac{z^\ell}{\sqrt{\pi\Gamma(\ell+1)}} \qquad (14)$$

leading to the scalar kernel

$$K_N(z, z') = \frac{1}{\pi}\, e^{-z\bar{z}/2}\, e^{-z'\bar{z}'/2} \sum_{\ell=0}^{N-1} \frac{(zz')^\ell}{\Gamma(\ell+1)}. \qquad (15)$$

Density of states.—Put into the integral form, the kernel (15) yields the finite–N density of states

$$R_1(z; N) = \frac{e^{-z\bar{z}}}{\pi\Gamma(N)} \int_0^\infty d\lambda\, e^{-\lambda}(\lambda + z\bar{z})^{N-1}. \qquad (16)$$

In terms of the upper incomplete gamma function

$$\Gamma(a, x) = \int_x^\infty dt\, t^{a-1} e^{-t} \qquad (17)$$

the level density equivalently reads

$$R_1(z; N) = \frac{\Gamma(N, z\bar{z})}{\pi\Gamma(N)}. \qquad (18)$$

A careful analysis of the integral (16) shows that, in the large–N limit, N complex eigenvalues are (almost) uniformly distributed within a circle of the radius $\sqrt{N}$ centered at $z = 0$,

$$R_1(z; N \gg 1) \simeq \pi^{-1}\theta(\sqrt{N} - |z|), \qquad (19)$$

$\theta(x)$ being a Heaviside step function.

In the vicinity $z_c = (\sqrt{N} + u)\, e^{i\varphi}$ of the edge $|z| = \sqrt{N}$ of the two-dimensional eigenvalue support described by (19), the density of states sharply crosses over from $R_1(z; N \gg 1) = \pi^{-1}$ at $|z| < \sqrt{N}$ to $R_1(z; N \gg 1) = 0$ at $|z| > \sqrt{N}$. The crossover is described by the local density of eigenvalues $R_1^{(\text{tails})}(u) = R_1(z_c; N)$ which, in the large–N limit, turns out to be independent of the matrix size N,

$$R_1^{(\text{tails})}(u) = \frac{1}{\pi\,(2\pi)^{1/2}} \int_{2u}^{\infty} dt\, e^{-t^2/2}. \tag{20}$$

Expressed in terms of the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} dt\, e^{-t^2} \tag{21}$$

the local density of states reads

$$R_1^{(\text{tails})}(u) = \frac{1}{2\pi}\, \mathrm{erfc}\left(u\sqrt{2}\right). \tag{22}$$

At $|u| \gg 1$, the tail asymptotics can be read out of (22) and are given by

$$R_1^{(\text{tails})}(u) \simeq \frac{1}{\pi} \left(\theta(-u) + \frac{e^{-2u^2}}{2(2\pi)^{1/2}\, u} \right). \tag{23}$$

Two-point correlation function.—At finite N, this simplest fluctuation characteristics is given by (13) and (15). In the large–N limit, when both z_1 and z_2 in (13) are situated inside the circle $|z| < \sqrt{N}$, the kernel (15) reduces to simple exponentials so that the two-point correlation function becomes N–independent,

$$R_2(z_1, z_2) = R_2(z_1, z_2; N \gg 1) = \frac{1}{\pi^2} \left(1 - e^{-|z_1 - z_2|^2}\right). \tag{24}$$

Make notice that the two-point correlation function $R_2(z_1, z_2; N)$ differs from the *density-density* correlation function $\hat{R}(z_1, z_2; N)$ defined by

$$\hat{R}(z_1, z_2; N) = \left\langle \mathrm{tr}\, \delta^2(z_1 - \mathcal{H})\ \mathrm{tr}\, \delta^2(z_2 - \mathcal{H}) \right\rangle_{\mathcal{H} \in \mathrm{GinUE}_N}. \tag{25}$$

The latter contains an additional δ–function contribution [38]

$$\hat{R}(z_1, z_2) = \hat{R}(z_1, z_2; N) = \frac{1}{\pi}\, \delta^2(z_1 - z_2) + R_2(z_1, z_2) \tag{26}$$

coming from the self-correlation of eigenlevels that chanced to meet in the complex plane.

3 Fermionic replica field theory

In this section we are going to re-derive the above nonperturbative results by mapping the Ginibre ensemble (6) of complex non-Hermitean random matrices onto a zero-dimensional (0D) *fermionic* replica field theory. Since the nonperturbative fluctuation formulae collected in Section 2 are well known for almost four decades (we remind that Ginibre's work [32] dates back to 1965), one may wonder why we should bother ourselves with such a minor issue. The answer prompted by the discussion in Section 1 is twofold. First, more than twenty years after their invention [3, 4], replica field theories largely remain unexplored territory from the viewpoint of their *controllable* treatment away from a perturbative sector. Second, learning intrinsic integrable structure of fermionic replica field theories in the simplest 0D limit – apart from encountering indisputable mathematical beauty of exact theory – creates a basis for future work beyond the RMT: Given an intimate connection [23] between integrability and the underlying physical symmetries, one may hope that some crucial characteristics of 0D replica partition functions are in fact not so peculiar to these simple models but remain true in a more general setting.

3.1 Replica partition functions

Density of states.—To determine the average density of complex eigenvalues of the matrix Hamiltonian $\mathcal{H} \in \mathrm{GinUE}_N$, we use a proper modification [22] of (4) and (5). A new recipe [22] is

$$R_1(z;N) = \lim_{n\to 0} \frac{1}{\pi n} \frac{\partial^2}{\partial z \partial \bar{z}} Z_n(z,\bar{z};N) \tag{27}$$

where the replica partition function $Z_n(z,\bar{z};N)$ is determined by the matrix integral

$$Z_{n\in\mathbb{R}}(z,\bar{z};N) = \left\langle \det{}^n(z-\mathcal{H}) \det{}^n(\bar{z}-\mathcal{H}^\dagger) \right\rangle_{\mathcal{H}\in\mathrm{GinUE}_N}. \tag{28}$$

Representing each of the determinants in (28) as a field integral over an N–component fermionic field and performing the averaging over $\mathcal{H} \in \mathrm{GinUE}_N$, one maps $Z_n(z,\bar{z};N)$ onto a fermionic replica sigma model, $Z_n(z,\bar{z};N) \mapsto \tilde{Z}_n(z,\bar{z};N)$, of the form [22]

$$\tilde{Z}_{n\in\mathbb{Z}^+}(z,\bar{z};N) = \left\langle \det{}^N \begin{pmatrix} z & -\mathcal{Q} \\ \mathcal{Q}^\dagger & \bar{z} \end{pmatrix} \right\rangle_{\mathcal{Q}\in\mathrm{GinU}_n}. \tag{29}$$

Importantly, while $Z_n(z, \bar{z}; N)$ in (28) is defined for arbitrary $n \in \mathbb{R}$, the replica parameter n in the representation (29) for the mapped replica partition function $\tilde{Z}_n(z, \bar{z}; N)$ is restricted – by derivation – to positive integers only, $n \in \mathbb{Z}^+$. To emphasise this difference between two types of partition functions, we will write either 'tilded' $\tilde{Z}_n$ ($n \in \mathbb{Z}^+$) or 'untilded' Z_n ($n \in \mathbb{R}$).

The matrix integral (29) can be reduced to an n–fold integral [22] by making use of a singular value decomposition of a complex matrix $\mathcal{Q} \in \mathbb{C}^{n\times n}$. Expressing it as $Q = U\Lambda V$ where $U \in \mathtt{U}(n)/\mathtt{U}(1)^n$, $V \in \mathtt{U}(n)$ and $\Lambda = \mathrm{diag}(\lambda_1^{1/2}, \cdots, \lambda_n^{1/2})$ with $\lambda_\ell \geq 0$, and calculating a Jacobian of the transformation $\mathcal{Q} \to (\Lambda, U, V)$, one derives [22]

$$\tilde{Z}_n(z, \bar{z}; N) = \int_0^\infty \prod_{\ell=1}^{n} d\lambda_\ell \, e^{-\lambda_\ell} \, (\lambda_\ell + z\bar{z})^N \prod_{\ell_1<\ell_2=1}^{n} |\lambda_{\ell_1} - \lambda_{\ell_2}|^2. \tag{30}$$

We reiterate that this representation makes sense for $n \in \mathbb{Z}^+$ only. This is precisely the reason why the replica limit (27) with Z_n replaced by $\tilde{Z}_n$ cannot be implemented directly.

Density-density correlation function.—Similarly to the level density (27), the density-density correlation function (25) can be retrieved from the replica limit

$$\hat{R}(z_1, z_2; N) = \lim_{n\to 0} \frac{1}{\pi^2 n^2} \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} Z_n(z_1, \bar{z}_1; z_2, \bar{z}_2; N) \tag{31}$$

where, for arbitrary $n \in \mathbb{R}$, the replica partition function $Z_n(z_1, \bar{z}_1; z_2, \bar{z}_2; N)$ is defined by

$$Z_{n\in\mathbb{R}}(z_1, \bar{z}_1; z_2, \bar{z}_2; N) = \\ \left\langle \det{}^n(z_1 - \mathcal{H}) \det{}^n(\bar{z}_1 - \mathcal{H}^\dagger) \det{}^n(z_2 - \mathcal{H}) \det{}^n(\bar{z}_2 - \mathcal{H}^\dagger) \right\rangle_{\mathcal{H}\in\mathrm{GinUE}_N}. \tag{32}$$

Upon a fermionic mapping [22] which restricts the replica parameter to $n \in \mathbb{Z}^+$, this generating function can be rewritten as

$$\tilde{Z}_{n\in\mathbb{Z}^+}(z_1, \bar{z}_1; z_2, \bar{z}_2; N) = \left\langle \det{}^N \left(\begin{array}{cc} \mathcal{Z} & -\mathcal{Q} \\ \mathcal{Q}^\dagger & \bar{\mathcal{Z}} \end{array} \right) \right\rangle_{\mathcal{Q}\in\mathrm{GinUE}_{2n}} \tag{33}$$

with the diagonal matrix $\mathcal{Z} = \mathrm{diag}(z_1 \, 1_n, \; z_2 \, 1_n)$. Note a tilde in (33).

For z_1, z_2 finite and of order unity, $\tilde{Z}_{n\in\mathbb{Z}^+}$ can be reduced [22] to a matrix integral over $U \in \mathtt{U}(2n)$ which eventually boils down to the n–fold integral[39]

$$\tilde{Z}_n(z_1,\bar{z}_1;z_2,\bar{z}_2;N) = e^{-2n(N-z\bar{z})} \int_{-1}^{+1} \prod_{\ell=1}^{n} d\lambda_\ell \, e^{(\omega\bar{\omega}/2)\lambda_\ell} \prod_{\ell_1<\ell_2=1}^{n} |\lambda_{\ell_1}-\lambda_{\ell_2}|^2. \tag{34}$$

Here,

$$z = \frac{z_1+z_2}{2}, \;\; \omega = z_1 - z_2. \tag{35}$$

As is the case (30), this representation makes sense for $n \in \mathbb{Z}^+$ so that the replica limit (31) with Z_n replaced by $\tilde{Z}_n$ cannot be implemented directly.

3.2 Replica partition functions as members of a positive Toda Lattice Hierarchy

By derivation, the n–fold integral representations (30) and (34) of the replica partition functions $\tilde{Z}_n(z,\bar{z};N)$ and $\tilde{Z}_n(z_1,\bar{z}_1;z_2,\bar{z}_2;N)$ stay valid for $n \in \mathbb{Z}^+$ only. Therefore, any attempt to retrieve spectral fluctuation properties of the matrix Hamiltonian $\mathcal{H}$ out of the replica limits (27) and (31) with Z_n replaced by $\tilde{Z}_n$ will inevitably face the problem of analytic continuation [40] of $\tilde{Z}_n$'s away from n positive integers. For this procedure to be controlled, an *exact* result for $\tilde{Z}_n$ is desired to start with.

For approximate treatment [9, 12] of the above replica partition functions the reader is referred to Ref. [22]. In this subsection we wish to explore another route which rests on *exact* and, therefore, truly nonperturbative evaluation of replica partition functions. A connection between nonlinear replica σ models and the theory of integrable hierarchies is at the heart of our formalism [23]. A proof that the nonperturbative fermionic replica partition functions form a positive, semi-infinite Toda Lattice Hierarchy is the first important outcome of our approach.

Bulk density of states.—To show how the Toda Lattice Hierarchy emerges in the context of (30), we represent the Vandermonde determinant there as

$$\prod_{\ell_1<\ell_2=1}^{n} (\lambda_{\ell_1}-\lambda_{\ell_2}) = \det(\lambda_k^{\ell-1})_{k,\ell=1,\cdots,n}, \tag{36}$$

simultaneously shift all λ_ℓ's therein by $z\bar{z}$, and perform the n–fold integral

(30) by means of the Andréief–de Bruijn integration formula [41, 42]

$$\int \prod_{\ell=1}^{n} d\mu(\lambda_\ell)\ \det[A_k(\lambda_\ell)]_{k,\ell=1,\cdots,n} \det[B_k(\lambda_\ell)]_{k,\ell=1,\cdots,n} = \\ n!\ \det\left(\int d\mu(\lambda) A_k(\lambda)\, B_\ell(\lambda)\right)_{k,\ell=1,\cdots,n} \tag{37}$$

which holds for any benign integration measure $d\mu(\lambda)$ given convergence of the integrals involved. Up to (irrelevant for our purposes) factorial prefactor, the interim result is

$$\tilde{Z}_n(z,\bar{z};N) = \det\left(\int_0^\infty d\lambda\, e^{-\lambda}(\lambda + z\bar{z})^{N+k+\ell}\right)_{k,\ell=0,\cdots,n-1}. \tag{38}$$

While exhibiting some beauty (in particular, $e^{-z\bar{z}}\tilde{Z}_1(z,\bar{z};N)$ coincides, up to a normalisation prefactor, with the density of states $R_1(z;N+1)$ in GinUE_{N+1}, see (16)), the representation (38) is not very informative or helpful. What is more helpful is another though totally equivalent form of (38),

$$\tilde{Z}_n(z,\bar{z};N) = e^{nz\bar{z}}(z\bar{z})^{n(n+N)}\,\tilde{\tau}_n(z\bar{z};N), \tag{39}$$

which involves the *Hankel determinant*

$$\tilde{\tau}_n(z\bar{z};N) = \det\left[\partial_{(z\bar{z})}^{k+\ell}\tilde{\tau}_1(z\bar{z};N)\right]_{k,\ell=0,\cdots,n-1} \tag{40}$$

with

$$\tilde{\tau}_0(z\bar{z};N) = 1, \quad \tilde{\tau}_1(z\bar{z};N) = \int_1^\infty d\lambda\,\lambda^N e^{-z\bar{z}\lambda} = \frac{\Gamma(N+1,z\bar{z})}{(z\bar{z})^{N+1}}. \tag{41}$$

Here, $\Gamma(a,x)$ is the upper incomplete gamma function (17). The initial condition for $\tilde{\tau}_0(z\bar{z};N)$ reflects the fact that $\tilde{Z}_0(z,\bar{z};N) = Z_0(z,\bar{z};N) = 1$, see (28).

The Hankel determinant (40) is a remarkable object. Whatever the function $\tilde{\tau}_1(z\bar{z};N)$ is, by virtue of the Darboux Theorem [43], the entire sequence $\{\tilde{\tau}_{k\in\mathbb{Z}^+}\}$ satisfies the equation

$$\tilde{\tau}_n\,\tilde{\tau}_n'' - (\tilde{\tau}_n')^2 = \tilde{\tau}_{n-1}\tilde{\tau}_{n+1}, \quad n \in \mathbb{Z}^+ \tag{42}$$

where the prime $'$ stands for $\partial_{(z\bar{z})}$. Equations (39) and (42) taken together with the known initial conditions $\tilde{\tau}_0 = 1$ and $\tilde{\tau}_1$ given by (41) establish [23] a hierarchy between nonperturbative fermionic replica partition functions $\tilde{Z}_n$ with different $n \in \mathbb{Z}^+$. The *exact* result (39), (41), and (42) is an alternative to an *approximate* positive-integer–n treatment of the very same replica partition function presented in Ref. [22].

Equation (42), *known as positive, semi-infinite Toda Lattice equation in the theory of integrable hierarchies* [24], *is the first indication of exact solvability hidden in replica field theories.* Importantly, emergence of the Toda Lattice Hierarchy is eventually due to the $\beta = 2$ Dyson's symmetry of the fermionic replica field theory encoded into the squared Vandermonde determinant in (30).

Density of states at the edge.—The above symmetry argument ensures that the replica partition function for the edge density of states in the GinUE$_N$ will obey the same Toda Lattice equation albeit with different initial conditions. Close to the edge $|z| = \sqrt{N}$ (see discussion next to (19)), one is interested in the large–N replica partition function $\tilde{Z}_n(z, \bar{z}; N)$ taken at $z = (\sqrt{N} + u)\, e^{i\varphi}$ in the regime $|u| \ll \sqrt{N}$. The latter, denoted as $\tilde{Z}_n^{(\text{tails})}(u)$, is independent of the matrix size N and equals

$$\tilde{Z}_n^{(\text{tails})}(u) = \int_0^\infty \prod_{\ell=1}^n dt_\ell \, e^{-t_\ell^2/2 - 2ut_\ell} \prod_{\ell_1 < \ell_2 = 1}^n |t_{\ell_1} - t_{\ell_2}|^2. \tag{43}$$

Irrelevant numeric prefactors were omitted.

Much in line with previous calculations, the n–fold integral (43) can be transformed into the Hankel determinant form

$$\tilde{Z}_n^{(\text{tails})}(u) = \det \left[\partial_u^{k+\ell} \tilde{Z}_1^{(\text{tails})}(u) \right]_{k,\ell=0,\cdots,n-1} \tag{44}$$

with

$$\tilde{Z}_0^{(\text{tails})}(u) = 1, \qquad \tilde{Z}_1^{(\text{tails})}(u) = e^{2u^2} \int_{2u}^\infty dt \, e^{-t^2/2} = \sqrt{\frac{\pi}{2}} \, e^{2u^2} \text{erfc} \left(u\sqrt{2} \right). \tag{45}$$

As soon as $\tilde{Z}_0^{(\text{tails})}(u) = 1$ (normalisation), the Darboux Theorem [43] can be applied to conclude that the entire sequence $\{\tilde{Z}_{k\in\mathbb{Z}^+}^{(\text{tails})}\}$ of replica partition functions at the edge of the two-dimensional eigenvalue support belongs to the positive, semi-infinite Toda Lattice Hierarchy

$$\tilde{Z}_n^{(\text{tails})} \, \tilde{Z}_n^{(\text{tails})\,\prime\prime} - (\tilde{Z}_n^{(\text{tails})\,\prime})^2 = \tilde{Z}_{n-1}^{(\text{tails})} \, \tilde{Z}_{n+1}^{(\text{tails})}, \quad n \in \mathbb{Z}^+ \tag{46}$$

the prime $'$ stands for ∂_u. Also, similarly to the observation made below (38), we notice that, up to a prefactor, $e^{-2u^2} \tilde{Z}_1^{(\text{tails})}(u)$ coincides with the edge density of states $R_1^{(\mathrm{tails})}(u)$ as given by (20). We will comment on this later on.

Bulk eigenvalue correlations.—By the same token, the replica partition function (34) designed to calculate the density-density correlation function via the replica limit (31) belongs to a Toda Lattice Hierarchy, too. Separating z and ω–dependent pieces in (34), one derives

$$\tilde{Z}_n(z_1, \bar{z}_1; z_2, \bar{z}_2; N) \quad = \quad e^{-2n(N-z\bar{z})}\tilde{Z}_n(\omega\bar{\omega}) \tag{47}$$

where $\tilde{Z}_n(\omega\bar{\omega})$ determined by the n–fold integral in (34) can be cast into the Hankel determinant form [13, 23]

$$\tilde{Z}_n(\omega\bar{\omega}) = \det\left(\partial_{(\omega\bar{\omega})}^{k+\ell}\,\frac{\sinh(\omega\bar{\omega}/2)}{\omega\bar{\omega}/2}\right)_{k,\ell=0,\cdots,n-1}. \tag{48}$$

The positive, semi-infinite Toda Lattice equation for $\{\tilde{Z}_{k\in\mathbb{Z}^+}\}$ readily follows by virtue of the Darboux Theorem.

3.3 Replica partition functions and Painlevé transcendents

While important for revealing integrability of the field theory, the Toda Lattice equation for *fermionic* replica partition functions $\tilde{Z}_{n\in\mathbb{Z}^+}$ is not much helpful in performing the replica limit, if taken *alone*. Indeed, a positive Toda Lattice equation gives no close expression for $\tilde{Z}_n$ as a function of $n \in \mathbb{Z}^+$ that would facilitate an analytic continuation to $n \in \mathbb{R}$ in general, and to the region $0 \le n \ll 1$ in particular. As was recently shown by Splittorff and Verbaarschot [29, 30], this difficulty can be circumvented if one succeeds in gaining a complementary hierarchical information about *bosonic* replica partition function. (The latter satisfies a negative [7] Toda Lattice equation). Viable yet surprisingly efficient [30] for random matrix models describing interactionless stochastic systems, this route is certainly unavailable for replica description of physical systems in presence of interaction [44] which requires to use *either* bosonic *or* fermionic field integrals in order to properly accommodate quantum statistics of interacting species.

Considering exact replica treatment of disordered interacting systems as a legitimate goal, it would be conceptually important to not rely on such a complementary information. Fortunately, for 0D interactionless systems at hand, it is indeed possible. Miraculously, the same Toda lattice equation governs the behaviour of so-called τ-functions arising in the Hamiltonian formulation [25-28] of the six Painlevé transcendents (PI – PVI), which are yet another fundamental object in the theory of nonlinear integrable systems.

Luckily, the Painlevé equations being second order nonlinear differential equations contain the hierarchy (or replica) index n as a *parameter*. As will be demonstrated below, this feature of Painlevé equations makes them serve as a proper starting point for constructing a consistent analytic continuation of nonperturbative replica partition functions away from n integers. *This Painlevé reduction further confirms exact solvability of replica σ models and assists [23] performing the replica limit.*

Bulk density of states.—Certainly being an option, the aforementioned Toda to Painlevé reduction [45] is not the only way to arrive at the sought Painlevé representation of the replica partition function $\tilde{Z}_n(z,\bar{z};N)$. An alternative approach would rest on the observation that the n–fold integral (30) is essentially a *Fredholm determinant* [46] associated with a gap formation probability

$$\mathbb{E}_n^{(0,z\bar{z})}(0;a) = \int_{z\bar{z}}^{\infty} \prod_{\ell=1}^{n} d\lambda_\ell \, e^{-\lambda_\ell} \, \lambda_\ell^a \prod_{\ell_1<\ell_2=1}^{n} |\lambda_{\ell_1} - \lambda_{\ell_2}|^2 \tag{49}$$

within the interval $(0, z\bar{z})$ in the spectrum of an auxiliary $n \times n$ Laguerre unitary ensemble. Celebrated result [47] due to Tracy and Widom states that

$$\mathbb{E}_n^{(0,z\bar{z})}(0;a) = \exp\left(\int_0^{z\bar{z}} dt \, \frac{\sigma_{\rm V}(t)}{t} \right) \tag{50}$$

where $\sigma_{\rm V}(t) = \sigma_n(t;a)$ is the fifth Painlevé transcendent satisfying the Jimbo-Miwa-Okamoto form of the Painlevé V equation [26, 48]

$$(t\sigma_{\rm V}'')^2 - (a\sigma_{\rm V}')^2 - (\sigma_{\rm V} - t\sigma_{\rm V}') \left[\sigma_{\rm V} - t\sigma_{\rm V}' + 4\sigma_{\rm V}' \left(\sigma_{\rm V}' + n + \frac{a}{2} \right) \right] = 0 \tag{51}$$

supplemented by the boundary condition [50, 51]

$$\sigma_{\rm V}(t)|_{t\to+\infty} \sim -nt + an - \frac{an^2}{t} + \mathcal{O}(t^{-2}). \tag{52}$$

We, thus, derive an exact Painlevé V representation of the replica partition function $\tilde{Z}_n(z,\bar{z};N)$ in the form

$$\tilde{Z}_n(z,\bar{z};N) = e^{nz\bar{z}} \exp\left(\int_0^{z\bar{z}} dt \, \frac{\sigma_{\rm V}(t)}{t} \right). \tag{53}$$

where $\sigma_{\rm V}(t) = \sigma_n(t; a = N)$.

Note that (51) – (53) contain the replica index $n \in \mathbb{Z}^+$ as a *parameter*. Taken together with the fact that the above Painlevé representation encodes *all* hierarchical inter-relations between the replica partition functions with various replica indices, it is very tempting to conjecture that (53), as it stands, holds beyond $n \in \mathbb{Z}^+$ as well. Indeed, for $n > -1$, this conclusion has recently been proven [49] by appealing to the Okamoto τ-function theory [26] of the fifth Painlevé transcendent. Therefore, the replica limit (27) with Z_n substituted by $\tilde{Z}_n$ can safely be implemented.

To proceed, we expand a solution to (51) around $n = 0$. Owing to the normalisation $\tilde{Z}_0 = 1$, the expansion starts with a term linear in n,

$$\sigma_{\mathrm{V}}(t) = \sigma_n(t; a) = \sum_{p=1}^{\infty} n^p f_p(t; a). \tag{54}$$

Only the first term of the above series is of our interest since the replica limit relates the bulk density of states $R_1(z; N)$ to the function $f_1(t; N)$ as

$$R_1(z; N) = \pi^{-1}[1 + f_1'(z\bar{z}; N)]. \tag{55}$$

Here $f_1(t; N)$ satisfies the differential equation

$$(t f_1'')^2 - (f_1 - t f_1')^2 - 2N f_1'(f_1 - t f_1') - (N f_1')^2 = 0 \tag{56}$$

subject to the conservation constraint [51]

$$\int_{\mathbb{C}} d^2 z\, R_1(z; N) = \int_0^{\infty} dt\, [1 + f_1'(t; N)] = N. \tag{57}$$

Identifying a complete square in (56), we reduce the latter to the Kummer differential equation

$$f_1 + (N - t) f_1' \pm t f_1'' = 0. \tag{58}$$

The constraint (57) makes us look for those solutions $f_1(t; N)$ whose first derivative f_1' is bounded at $t = +\infty$ and possibly has an integrable singularity at $t = +0$. This class of functions sought welcomes the sign $(-)$ in (58) leading to a general solution

$$f_1(t; N) = C_1\,(N - t) + C_2\, t^{N+1}\, {}_1F_1(N, N+2; -t). \tag{59}$$

The constraint (57) uniquely fixes unknown constants C_1 and C_2 be $C_1 = 0$ and $C_2 = -1/\Gamma(N+2)$. This yields [52]

$$f_1(t; N) = -\frac{t^{N+1}}{\Gamma(N+2)}\, {}_1F_1(N; N+2; -t) \tag{60}$$

where ${}_1F_1(a;b;t)$ is the confluent hypergeometric function of Kummer. Consequently, the replica limit (27) of the fermionic partition function (53) results in the density of states (55) of the form

$$R_1(z;N) = \frac{\Gamma(N, z\bar{z})}{\pi\Gamma(N)} \tag{61}$$

which is equivalent to (16) and (18). The small–n expansion of the fermionic replica partition function

$$\ln\, \tilde{Z}_n(z,\bar{z};N) = \\ n(z\bar{z})\left[1 + \frac{(z\bar{z})^N}{(N+1)\Gamma(N+2)}\; {}_2F_2(N, N+1; N+2, N+2; -z\bar{z})\right] + \mathcal{O}(n^2) \tag{62}$$

is behind the result (61). It should be stressed that the finite–N result (61) cannot be produced by approximate treatment [22] of replicas which heavily relies on availability of a large parameter ($N \gg 1$) in the integral representation (30).

Density of states at the edge.—With the replica partition function $\tilde{Z}_n^{(\rm tails)}$ given by (43), the density of states at the edge $|z| = \sqrt{N}$ of the two-dimensional eigenvalue support is determined by the replica limit

$$R_1^{(\rm tails)}(u) = \lim_{n\to 0} \frac{1}{\pi n} \frac{\partial^2}{\partial u^2}\, \tilde{Z}_n^{(\rm tails)}(u). \tag{63}$$

The partition function $\tilde{Z}_n^{(\rm tails)}$ can again be viewed as a Fredholm determinant associated with a gap formation probability

$$\mathbb{E}_n^{(u,\infty)}(0) = \int_{-\infty}^{u} \prod_{\ell=1}^{n} d\lambda_\ell\, e^{-\lambda_\ell^2} \prod_{\ell_1<\ell_2=1}^{n} |\lambda_{\ell_1} - \lambda_{\ell_2}|^2 \tag{64}$$

within the interval (u,∞) in the spectrum of an auxiliary $n\times n$ Gaussian Unitary Ensemble. In terms of the fourth Painlevé transcendent [27] $\sigma_{\rm IV}$, it reads [46, 53]

$$\mathbb{E}_n^{(u,\infty)}(0) = \mathbb{E}_n^{(0,\infty)}(0) \exp\left(\int_0^u dt\, \sigma_{\rm IV}(t)\right) \tag{65}$$

where $\sigma_{\rm IV}(t) = \sigma_n(t; a=0)$ satisfies the Painlevé IV equation in the Jimbo-Miwa-Okamoto form

$$(\sigma_{\rm IV}'')^2 - 4(t\sigma_{\rm IV}' - \sigma_{\rm IV})^2 + 4\sigma_{\rm IV}'(\sigma_{\rm IV}' - 2a)(\sigma_{\rm IV}' + 2n) = 0 \tag{66}$$

subject to the boundary condition [51]

$$\sigma_{\rm IV}(t)|_{t\to-\infty} \sim -2nt - \frac{n(a+n)}{t} + \mathcal{O}(t^{-3}). \tag{67}$$

In both (66) and (67) the parameter a has to be set to zero. The above equations result in the Painlevé IV representation of the fermionic replica partition function

$$\tilde{Z}_n^{(\rm tails)}(u) = e^{2nu^2}\,\exp\left(\int_0^{-u\sqrt{2}} dt\,\sigma_{\rm IV}(t)\right) \tag{68}$$

which holds for $n \in \mathbb{Z}^+$.

To perform the replica limit (63), we follow the technology that led us to the small–n expansion (62). To this end we have to assume [54] that the Painlevé IV representation (68) stays valid in a vicinity of $n=0$. Writing down [55]

$$\sigma_{\rm IV}(t) = \sigma_n(t;a) = \sum_{p=1}^{\infty} n^p g_p(t;a), \tag{69}$$

one derives from here, (63) and (68) that

$$R_1^{(\rm tails)}(u) = \frac{4}{\pi}\left[1 + \frac{1}{2}\,g_1^\prime(-u\sqrt{2};0)\right]. \tag{70}$$

Equation for $g_1 = g_1(t; a=0)$ follows from (66) and (69),

$$g_1^{\prime\prime} \pm 2(tg_1^\prime - g_1) = 0 \tag{71}$$

while the boundary conditions are [56]

$$\begin{array}{rl} g_1^\prime(-\infty) + 2 = 0 & \quad \text{(convergence)} \\ g_1^\prime(+\infty) + 3/2 = 0 & \quad \text{(conservation)} \end{array} \tag{72}$$

The sign $(-)$ in (71) leads to a solution with unbounded first derivative $g_1^\prime(t)$ at both infinities and is therefore incompatible with (72). Equation (71) with the sign $(+)$ yields a general solution

$$g_1(t) = C_1\,t + C_2\left(t\,{\rm erf}\,t + \frac{1}{\sqrt{\pi}}\,e^{-t^2}\right). \tag{73}$$

Boundary conditions (72) fix the constants [57] be $C_1 = -7/4$ and $C_2 = 1/4$. By virtue of (70) this results in the density of states

$$R_1^{(\text{tails})}(u) = \frac{1}{2\pi} \operatorname{erfc}\left(u\sqrt{2}\right). \tag{74}$$

This is identically equivalent to (23). The small–n expansion of the fermionic replica partition function

$$\ln \tilde{Z}_n^{(\text{tails})}(u) = \frac{n}{4}\left[u^2 - \frac{1}{\sqrt{2\pi}}\, u e^{-2u^2} - \left(u^2 + \frac{1}{4}\right) \operatorname{erf}\left(u\sqrt{2}\right)\right] + \mathcal{O}(n^2) \tag{75}$$

is behind the result (74). Being exact, the formula (74) describes the tails of level density *both* inside ($u < 0$) and outside ($u > 0$) of the circle [32] $|z| = \sqrt{N}$. The approximate treatment [22] of replicas has failed to reproduce the density of states outside the circle, $|z| > \sqrt{N}$.

Density-density correlation function. To determine this spectral characteristics, we put the replica limit (31) into the form

$$\hat{R}(z_1, z_2) = \lim_{n\to 0} \frac{1}{\pi^2 n^2} \left(\frac{1}{4}\,\partial_z^2 - \partial_\omega^2\right)\left(\frac{1}{4}\,\partial_{\bar z}^2 - \partial_{\bar\omega}^2\right) \tilde{\Upsilon}_n(z, \bar z; \omega, \bar\omega) \tag{76}$$

involving the variables z and ω as defined by (35). The notation $\tilde{\Upsilon}_n(z, \bar z; \omega, \bar\omega)$ stands for the replica partition function [39]

$$\tilde{\Upsilon}_n(z, \bar z; \omega, \bar\omega) = \tilde{Z}_n\left(z + \frac{\omega}{2}, \bar z + \frac{\bar\omega}{2}; z - \frac{\omega}{2}, \bar z - \frac{\bar\omega}{2}\right). \tag{77}$$

At $z \pm \omega/2$ of order unity, the n–fold integral representation (34) makes it possible to express $\tilde{\Upsilon}_n$ in terms of a gap formation probability

$$\mathbb{E}_n^{(\omega\bar\omega,\infty)}(0; a) = \int_0^{\omega\bar\omega} \prod_{\ell=1}^{n} d\lambda_\ell\, e^{-\lambda_\ell}\, \lambda_\ell^a \prod_{\ell_1<\ell_2=1}^{n} |\lambda_{\ell_1} - \lambda_{\ell_2}|^2 \tag{78}$$

within the interval $(\omega\bar\omega, \infty)$ in the spectrum of an auxiliary $n \times n$ Laguerre unitary ensemble (compare to (49)). The result due to Tracy and Widom [47] states that

$$\mathbb{E}_n^{(\omega\bar\omega,\infty)}(0, 0) = \exp\left(-\int_{\omega\bar\omega}^{\infty} dt\, \frac{\sigma_{\rm V}(t)}{t}\right) \tag{79}$$

where the fifth Painlevé transcendent $\sigma_{\rm V}(t) = \sigma_n(t; a = 0)$ satisfies the equation (51) with $a = 0$ and meets the boundary condition [58]

$$\sigma_{\rm V}(t)\big|_{t\to\infty} \sim \frac{t^{2n-1}}{\Gamma^2(n)}\, e^{-t}\left(1 + \mathcal{O}(t^{-1})\right). \tag{80}$$

The above equations yield, for $n \in \mathbb{Z}^+$, the exact representation

$$\tilde{\Upsilon}_n(z,\bar{z};\omega,\bar{\omega}) = \frac{\exp\left[n(2z\bar{z}+\omega\bar{\omega}/2)\right]}{(\omega\bar{\omega})^{n^2}} \exp\left(-\int_{\omega\bar{\omega}}^{\infty} dt\, \frac{\sigma_{\rm V}(t)}{t}\right). \tag{81}$$

To implement the replica limit, we have to analytically continue (81) into a vicinity of $n=0$. Although at the moment we do not have a proof that (81) as it stands also holds for n away from positive integers, armed with the previous experience we are going to conjecture that this is indeed the case so that

$$\hat{R}(z_1,z_2) = \frac{1}{2\pi^2} + \frac{1}{\pi^2}\lim_{n\to 0}\frac{1}{n^2}\left(\partial_\omega\partial_{\bar{\omega}}\right)^2 \frac{\exp(n\omega\bar{\omega}/2)}{(\omega\bar{\omega})^{n^2}} \exp\left(-\int_{\omega\bar{\omega}}^{\infty} dt\, \frac{\sigma_{\rm V}(t)}{t}\right). \tag{82}$$

Equation (82) suggests that the two functions, $h_1(t)$ and $h_2(t)$, of the small–n expansion

$$\sigma_{\rm V}(t) = \sum_{p=1}^{\infty} n^p\, h_p(t) \tag{83}$$

contribute the density-density correlation function in the replica limit. As we tend to avoid explicit reference to the boundary conditions for Painlevé transcendents, the easiest way to determine h_1 is to notice that, at $|z| \ll \sqrt{N}$, the bulk density of states equals

$$R_1(z;N) = \lim_{n\to 0}\frac{1}{\pi n}\frac{\partial^2}{\partial z\partial\bar{z}}\,\tilde{Z}_n(z,\bar{z};0,0;N). \tag{84}$$

This is so because, at $n \to 0$, the partition functions (28) and (32) taken at $z_1 = z$ and $z_2 = 0$ become indistinguishable if considered as functions of the energy variable z. Given (77) and (81), one derives

$$\tilde{Z}_n(z,\bar{z};0,0;N) = \frac{\exp(nz\bar{z})}{(z\bar{z})^{n^2}} \exp\left(-\int_{z\bar{z}}^{\infty} dt\, \frac{\sigma_{\rm V}(t)}{t}\right). \tag{85}$$

Only linear in n term of the expansion (83) contributes the replica limit (84) yielding

$$R_1(z) = \frac{1+h_1'(z\bar{z})}{\pi}. \tag{86}$$

According to the replica result (61), this must be equal to $1/\pi$ whence we conclude that the function $h_1(t)$ is a constant. Further, the equation

$$(th_1'')^2 - (h_1 - th_1')^2 = 0 \tag{87}$$

following from (51) and (83) sets

$$h_1(t) = 0. \tag{88}$$

Therefore, the first nontrivial term in the expansion (83) actually starts with $n^2 h_2(t)$ where $h_2(t)$ satisfies the equation

$$t h_2'' \pm (t h_2' - h_2) = 0. \tag{89}$$

The sign $(-)$ is the one that meets existence arguments applied to (79). As a result, we come down to

$$h_2(t) = C_1 t + C_2 \, E_2(t) \tag{90}$$

where $E_2(t)$ is the exponential integral

$$E_n(z) = \int_1^\infty dt \, \frac{e^{-zt}}{t^n}, \quad \mathrm{Re}\, z > 0. \tag{91}$$

Again, by existence arguments, C_1 must be set to zero to ensure convergence of the integral in the exponent of (79). To fix the constant C_2, we make use of the observation that $\tilde{\Upsilon}_n$ has to be finite at $\omega = 0$. This brings $C_2 = 1$ so that [59]

$$h_2(t) = E_2(t). \tag{92}$$

Collecting (81), (83), (88) and (92), we end up with the following nonperturbative small–n expansion [23] of the logarithm of the replica partition function:

$$\begin{aligned} \ln \tilde{\Upsilon}_n(z, \bar{z}; \omega, \bar{\omega}) = n \left(2 z \bar{z} + \frac{\omega \bar{\omega}}{2} \right) - \\ n^2 \big[\ln (\omega \bar{\omega}) + E_1(\omega \bar{\omega}) - E_2(\omega \bar{\omega}) \big] + \mathcal{O}(n^3). \end{aligned} \tag{93}$$

The replica limit (76) applied to (93) culminates in the exact result for the density-density correlation function

$$\hat{R}(z_1, z_2) = \frac{1}{\pi} \delta^2(z_1 - z_2) + \frac{1}{\pi^2} \left(1 - e^{-|z_1 - z_2|^2} \right). \tag{94}$$

Notice a presence of the δ–functional contribution in (94) describing the self-correlation of complex eigenlevels. The latter is inaccessible by the approximate treatment [22] of replicas.

3.4 Comment on a puzzle [22]

In the paper [22], Nishigaki and Kamenev have noticed that there exists a 'striking resemblance' between the finite–N densities of states and the replica partition functions for non-Hermitean random matrices belonging to all three universality classes $\beta = 1, 2$ and 4. In the context of the present study, the observation of the authors of Ref. [22] can be translated into the identity (see two remarks below (38) and (46), respectively)

$$R_1(z; N) \propto e^{-z\bar{z}}\, \tilde{Z}_1(z, \bar{z}; N-1) \tag{95}$$

that links the density of states $R_1(z; N)$ in GinUE_N to the replica partition function $\tilde{Z}_1(z, \bar{z}; N-1)$ for the same ensemble albeit of a smaller dimension GinUE_{N-1}.

The identity (95) is not a miracle and can well be understood as a consequence of the relation (12). Indeed, viewing the scalar kernel $K_N(z_1, z_2)$ in (12) as a matrix integral [60, 61]

$$K_N(z_1, z_2) \propto e^{-z_1\bar{z}_1/2} e^{-z_2\bar{z}_2/2} \left\langle \det\left(z_1 - \mathcal{H}\right)\, \det\left(z_2 - \mathcal{H}^\dagger\right) \right\rangle_{\mathcal{H}\in \mathrm{GinUE}_{N-1}}$$

one readily identifies

$$K_N(z, \bar{z}) \propto e^{-z\bar{z}}\, \tilde{Z}_1(z, \bar{z}; N-1) \tag{96}$$

whence (95) follows [62]. Interestingly, so explained identity (95) taken together with (27) leads, in the context of fermionic replica field theory, to a much less trivial statement

$$\frac{\partial^2}{\partial z \partial \bar{z}}\, \tilde{Z}_n(z, \bar{z}; N) = \pi n\, e^{-z\bar{z}} \tilde{Z}_1(z, \bar{z}; N-1) + \mathcal{O}(n^2). \tag{97}$$

Similar small–n expansions should exist for two other ($\beta = 1$ and 4) universality classes in non-Hermitean RMT.

4 Discussion

In the present paper we have offered a detailed account of a nonperturbative approach [23] to zero-dimensional fermionic replica field theories which is based on exact representation of replica partition functions in terms of Painlevé transcendents. Focusing on Ginibre ensemble [32] of complex non-Hermitean random matrices, we have revealed an intrinsic integrability of associated replica field theories. It materialises in two ways: First, at $n \in \mathbb{Z}^+$, the replica partition functions were proven to belong to a positive, semi-infinite Toda Lattice Hierarchy. Second, the very same replica partition functions were shown to be expressible in terms of solutions to Painlevé equations which (i) contain the replica index as a single *parameter* and which (ii) implicitly encode *all* hierarchical inter-relations between the fermionic replica partition functions with various replica indices.

The above two observations [(i) and (ii)] led us to conjecture that Painlevé representations of fermionic replica partition functions stay valid beyond $n \in \mathbb{Z}^+$ and, in particular, in a vicinity of $n = 0$. Indeed, for a particular case of the replica partition function $\tilde{Z}_n(z, \bar{z}; N)$ designed to determine the bulk density of complex eigenvalues, this conjecture was rigorously proven [49] by appealing to the Okamoto τ-function theory of the Painlevé V. (Similar, in spirit, proof was previously given [23] in the context of the one-point Green function in the finite–N Gaussian Unitary Ensemble where a Painlevé IV equation arises). Once justified, it is no surprise that taking the replica limit of the Painlevé-represented fermionic partition function $\tilde{Z}_n(z, \bar{z}; N)$ has culminated in reproducing exact nonperturbative results for the bulk density of states. In other cases, which include the tails of level density and the density-density correlation function in the spectrum bulk, although implemented without a formal justification, the replica limit of replica partition functions expressed in terms of Painlevé transcendents has also brought exact nonperturbative results. This fact as well as other encouraging applications [23] of the present method make us look further into the rational reasons behind its success. Possibly, recent developments [63] in the field of extended Toda Hierarchy may give us the right lead.

Finally, a remark is in order aimed to pinpoint the difference between the approach [23] detailed in this paper and a complementary approach recently elaborated in Refs. [29, 30]. Although both approaches exploit the very same replica representations of quantum correlation functions as a starting point, the two frameworks are conceptually different. The present approach [23, 7] based on exact Painlevé evaluation of *fermionic* replica partition functions followed by their continuation into a vicinity of $n = 0$ makes *no* reference whatsoever to *bosonic* partition functions. On the contrary, the approach [29, 30] exploiting the replica limit of the Toda Lattice equation for replica partition functions rests explicitly, and unavoidably, on the observation that suitably normalised fermionic and bosonic replica partition functions are the members of a *single* Toda Lattice Hierarchy albeit belonging to its positive (fermionic) and negative (bosonic) branches, respectively. Implemented on the level of such an infinite – *supersymmetric in essence* – Toda Lattice equation, the replica limit reveals a remarkable factorisation of quantum correlation functions for an interactionless matrix Hamiltonian into a product of *both* fermionic and bosonic partition functions. It is this factorisation which – in order to be materialised on the operational level – explicitly infuses [31, 64] a missing bosonic (or fermionic) information to what early appeared to be a pure fermionic (or bosonic) formulation of the field theory. While facilitating calculation of correlation functions in the interactionless case, this feature makes the approach [29, 30] be potentially inapplicable for a nonperturbative replica treatment [44] of 0D Hamiltonians with interactions [65-67] whose presence requires using of *either* bosonic *or* fermionic field integrals in order to accommodate a proper quantum statistics of interacting species.

I thank Peter Forrester, Alex Kamenev, and Jac Verbaarschot for clarifying correspondence and Joshua Feinberg, Shmuel Fishman and Ady Stern for helpful discussions. This work was supported in part by the Albert Einstein Minerva Centre for Theoretical Physics at the Weizmann Institute of Science.

References

[1] S.F. Edwards and P.W. Anderson, J. Phys. F: Met. Phys. **5**, 965 (1975).

[2] See (29) below for immediate example of a fermionic replica filed theory formulated in terms of some matrix field $\mathcal{Q} \in \mathbb{C}^{n\times n}$. Explanations will follow.

[3] F. Wegner, Z. Phys. B **35**, 207 (1979).

[4] L. Schäfer and F. Wegner, Z. Phys. B **38**, 113 (1980).

[5] K.B. Efetov, A.I. Larkin, and D.E. Khmelnitskii, Zh. Eksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].

[6] G. Parisi, e-print cond-mat/0207334 (2002).

[7] E. Kanzieper, *Replica sigma models and nonlinear integrable hierarchies*, unpublished (2003).

[8] J.J.M. Verbaarschot and M.R. Zirnbauer, J. Phys. A: Math. and Gen. **17**, 1093 (1985).

[9] A. Kamenev and M. Mézard, J. Phys. A: Math. and Gen. **32**, 4373 (1999).

[10] A. Kamenev and M. Mézard, Phys. Rev. B **60**, 3944 (1999).

[11] I.V. Yurkevich and I.V. Lerner, Phys. Rev. B **60**, 3955 (1999).

[12] M.R. Zirnbauer, e-print cond-mat/9903338 (1999).

[13] E. Kanzieper, Nucl. Phys. B **596**, 548 (2001).

[14] In the perturbative region of the field theory, the replica parameter n merely serves as a book-keeping tool identifying unphysical vacuum loops in a diagrammatic expansion for the partition function Z_n. Due to a finite number of expansion terms being taken into account, the dependence of Z_n on n is algebraic or rational at most making the analytic continuation away from $|n| \in \mathbb{Z}^+$ be straightforward.

[15] Recently formulated Keldysh field theory is a possible alternative. See: A. Kamenev and A. Andreev, Phys. Rev. B **60**, 3944 (1999).

[16] M.L. Mehta, *Random Matrices* (Academic Press, New York, 1991).

[17] K. Efetov, *Supersymmetry in Disorder and Chaos* (CUP, Cambridge, 1997).

[18] M. Mézard, G. Parisi, and M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).

[19] D.M. Gangardt and A. Kamenev, Nucl. Phys. B **610**, 578 (2001).

[20] S. M. Nishigaki, D. M. Gangardt, and A. Kamenev, e-print cond-mat/0207301 (2002).

[21] D. Dalmazi and J.J.M. Verbaarschot, Nucl. Phys. B **592**, 419 (2001).

[22] S.M. Nishigaki and A. Kamenev, J. Phys. A: Math. and Gen. **35**, 4571 (2002).

[23] E. Kanzieper, Phys. Rev. Lett. **89**, 250201 (2002).

[24] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices* (AMS, Providence, RI, 2000).

[25] K. Okamoto, Ann. Mat. Pura Appl. (4), **146**, 337 (1987).

[26] K. Okamoto, Japan. J. Math. **13**, 47 (1987).

[27] K. Okamoto, Math. Ann. **275**, 221 (1986).

[28] K. Okamoto, Funkcial. Ekvac. **30**, 305 (1987).

[29] K. Splittorff and J.J.M. Verbaarschot, Phys. Rev. Lett. **90**, 041601 (2003).

[30] K. Splittorff and J.J.M. Verbaarschot, e-print hep-th/0310271 (2003).

[31] In particular, *both* compact (fermionic) and non-compact (bosonic) integrals for partition functions appear explicitly in the formalism [30].

[32] J. Ginibre, J. Math. Phys. **6**, 440 (1965).

[33] M. Mineev-Weinstein, P.B. Wiegmann, and A. Zabrodin, Phys. Rev. Lett. **84**, 5106 (2000).

[34] O. Agam, E. Bettelheim, P. Wiegmann, and A. Zabrodin, Phys. Rev. Lett. **88**, 236801 (2002).

[35] A. Zabrodin, e-print cond-mat/0210331 (2002).

[36] Y.V. Fyodorov and H.-J. Sommers, J. Phys. A: Math. and Gen. **36**, 3303 (2003).

[37] The normalisation constant is fixed by the integration measure $\prod_{k,\,\ell=1}^{N} d\mathrm{Re}\, H_{k\ell}\, d\mathrm{Im}\, H_{k\ell}$.

[38] The δ–function in the complex plane is understood as $\delta^2(z) = \delta(\mathrm{Re}\, z)\, \delta(\mathrm{Im}\, z)$.

[39] In the regime in question, the parameter N enters $\tilde{Z}_n$ [see (34)] as a prefactor vanishing in the replica limit. For this reason, we will sometimes drop N from the arguments of $\tilde{Z}_n$. This is also the reason why N does not appear as a parameter in the l.h.s. of (76).

[40] Unless we know how to interprete [6] the integration measure at $n \notin \mathbb{Z}^+$.

[41] C. Andréief, Mém. de la Soc. Sci., Bordeaux **2**, 1 (1883).

[42] N.G. de Bruijn, J. Indian Math. Soc. **19**, 133 (1955).

[43] G. Darboux, *Lecons sur la Theorie Generale des Surfaces et les Applications Geometriques du Calcul Infinitesimal*, Vol. II: XIX (Chelsea, New York, 1972).

[44] A.M. Finkelstein, Zh. Éksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].

[45] P.J. Forrester and N.S. Witte, Commun. Pure Appl. Math. **55**, 679 (2002).

[46] C.A. Tracy and H. Widom, Commun. Math. Phys. **163**, 33 (1994).

[47] C.A. Tracy and H. Widom, Commun. Math. Phys. **161**, 289 (1994).

[48] M. Jimbo, T. Miwa,Y. Môri, and M. Sato, Physica D **1**, 80 (1980).

[49] E. Kanzieper, to appear in: *Progress in Field Theory Research* (Nova Science Publishers, New York, 2004).

[50] P. Forrester, *Log-Gases and Random Matrices*, Ch. 6 (book in progress, 2002).

[51] In fact, we do not actually need the boundary conditions [(52) or (67)]: general arguments based on existence, convergence and conservation laws do the job.

[52] One may verify that $\sigma_{\rm V}(t) = n\, f_1(t;N) + \cdots$ obeys the boundary condition (52) up to the terms linear in n:

$$\sigma_{\rm V}(t)|_{t\to+\infty} \sim -nt + nN.$$

[53] P.J. Forrester and N.S. Witte, Commun. Math. Phys. **219**, 357 (2001).

[54] This assumption might have been proven by examining (51) in the large–N scaling limit specified prior to (43).

[55] A term with $p = 0$ is forbidden by the normalisation $\tilde{Z}_0^{(\rm tails)} = 1$.

[56] Indeed, as the density of states $R_1^{(\rm tails)}(u)$ at the edge equals $R_1^{(\rm tails)}(u) = R_1(z = (\sqrt{N}+u)e^{i\varphi}; N)$, the conservation constraint $\int_{\mathbb{C}} d^2z\, R_1(z;N) = N$ translates to

$$\int_{-1}^{\infty} d\tau\, (1+\tau)\, R_1^{(\rm tails)}(\tau\sqrt{N}) = \frac{1}{2\pi}.$$

Taking the $N \to \infty$ limit and appealing to (70) yields the statement (72).

[57] Given $C_1 = -7/4$ and $C_2 = 1/4$, one may verify that $\sigma_{\rm IV}(t) = n\, g_1(t;0) + \cdots$ with $g_1(t)$ in the form (73) obeys the boundary condition (67) up to the term linear in n:

$$\sigma_{\rm IV}(t)|_{t\to-\infty} \sim -2nt. \qquad (98)$$

[58] J. Baik, Int. Math. Res. Not. **2002**, 1739 (2002).

[59] One may verify that so derived $\sigma_{\rm V}(t) = n^2 h_2(t) + \cdots$ meets the small–n version of the boundary condition (80):

$$\sigma_{\rm V}(t)|_{t\to\infty} \sim \frac{n^2}{t}\, e^{-t}.$$

[60] This representation can be proven along the lines of Ref. [61] where a similar matrix representation appears for a scalar kernel in the context of Hermitean random matrices.

[61] P. Zinn-Justin, Commun. Math. Phys. **194**, 631 (1998).

[62] Similarly, for the matrix Hamiltonian $\mathcal{H} \in {\rm GUE}_N$ taken from the Gaussian Unitary Ensemble, one easily derives the following relation

$$R_1(\epsilon; N) \propto e^{-\epsilon^2}\, \tilde{Z}_2(\epsilon; N-1)$$

between the density of states $R_1(\epsilon; N)$ in the ${\rm GUE}_N$ and the fermionic replica partition function $\tilde{Z}_2(\epsilon; N-1)$ for the ${\rm GUE}_{N-1}$.

[63] G. Carlet, B. Dubrovin, and Y. Zhang, e-print nlin.SI/0306060 (2003).

[64] I thank Jac Verbaarschot for pointing out to me that evaluation of non-compact integrals [31] can be traded for solving an appropriate Painlevé equation [23] taken at negative values of the replica index n. See, also, a remark below (21) in Ref. [29].

[65] I.L. Kurland, I.L. Aleiner, and B.L. Altshuler, Phys. Rev. B **62**, 14886 (2000).

[66] Y. Oreg, P.W. Brouwer, X. Waintal, and B.I. Halperin, e-print cond-mat/0109541 (2001).

[67] I.L. Aleiner, P.W. Brouwer, and L.I. Glazman, Phys. Rep. **358**, 309 (2002).