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A result on the phase diagram of a Ginzburg-Landau problem *

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Abstract

We study mathematically the Abrikosov [JETP Lett. **32**, 1174 (1957)] modelization of superconductors, which uses the Ginzburg-Landau phenomenological theory. We first prove the qualitative shape of the phase diagram, which is found in the physical literature. We then study in detail the special case, when the critical Ginzburg Landau parameter k is equal to $1/\sqrt{2}$. This allows us to prove that the critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = 1/\sqrt{2}$.

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1 Introduction

In 1950 V. Ginzburg and L. Landau [1] have proposed a model describing the various states of a superconducting material. They have introduced a functional depending on a *wave function* ϕ and a *magnetic potential vector* $\mathbf{A}$, whose local minima describe the properties of the material. In this model $|\phi|^2$ represents the local density of superconducting electrons [2 - 5].

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Abrikosov [6] has found a particular solution to the Ginzburg-Landau model, which predicts a periodic structure for the zeros of ϕ, which was subsequently observed in experiments. His model depends on two positive parameters k and H_{ext}, called *Ginzburg-Landau parameter* and *external magnetic field*, respectively. It also assumed that:

1. The superconductor is infinite, homogeneous and isotropic.
2. The magnetic field $\mathbf{H}_{\mathsf{ext}} = (0, 0, H_{\mathsf{ext}})$ is constant.
3. The energy functional $F(\phi, \mathbf{A})$ has a Ginzburg-Landau form and depends on the *Ginzburg-Landau parameter* k.
4. The pairs $(\phi, \mathbf{A})$ considered are gauge invariant along the z-axis and also along a lattice of $\mathbb{R}^2$.
5. The lattice has a fixed shape and there is one quantum flux per its unit cell.

After change of variable, as described in Section 2, we obtain the following formulation of the problem:

We define $\mathcal{L}$ as a lattice of $\mathbb{R}^2$ with fundamental domain Ω of area 1 and define the vector bundle E_1 over $\mathbb{R}^2/\mathcal{L}$ as the vector bundle, whose C^∞ sections are described by

$$C^\infty(E_1) = \left\{ \begin{array}{c} u : \mathbb{R}^2 \to \mathbb{C} \text{ s.t. } \forall (x,y) \in \mathbb{R}^2, \forall v = (v_x, v_y) \in \mathcal{L}, \\ u((x,y)+v) = e^{i\pi(v_x y - v_y x)} u(x,y) \end{array} \right\}.$$

The vector bundle E_1 is non-trivial; this implies that any section $u \in C^\infty(E_1)$ has at least one zero in $\mathbb{R}^2/\mathcal{L}$.

The potential vector $\mathbf{a}$ belongs to the space

$$\{\mathbf{a} \in H^1_{\mathsf{loc}}(\mathbb{R}^2; \mathbb{R}^2) \text{ such that } \operatorname{div} \mathbf{a} = 0,\ \mathbf{a} \text{ is } \mathcal{L}\text{-periodic and } \int_\Omega \mathbf{a} = 0\}.$$

We denote by $\mathcal{A}$ the space of all pairs $(u, \mathbf{a})$ with u being a H^1_{loc} section of E_1 and $\mathbf{a}$ belonging to the above space.

Denote H_{int} as the *internal magnetic field* and $E_{k,H_{\mathsf{int}}}$ the functional defined over $\mathcal{A}$ by

$$E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) \quad = \quad \int_\Omega \tfrac{\mu}{2} \| i\boldsymbol{\nabla} u + (\mathbf{A}_0 + \mathbf{a}) u \|^2 + \tfrac{1}{4}(1 - |u|^2)^2 + \tfrac{\mu^2 k^2}{2} |\operatorname{curl} \mathbf{a}|^2$$

with $\mu = \frac{H_{\mathsf{int}}}{2\pi k}$ and $\mathbf{A}_0 = \pi \begin{pmatrix} -y \\ x \end{pmatrix}$. We then define the energy of the superconductor as

$$E_{k,H_{\mathsf{ext}}}(H_{\mathsf{int}}, u, \mathbf{a}) = E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) + \frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2.$$

The term $\frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2$ is a simple magnetic energy, while the term $E_{k,H_{\mathsf{int}}}$ is the internal energy of the superconductor. The energy $\mathcal{E}_{k,H_{\mathsf{ext}}}$ is then defined as the minimum of $E_{k,H_{\mathsf{ext}}}$ over all magnetic fields H_{int} and pairs $(u, \mathbf{a}) \in \mathcal{A}$. Also we denote $m_E(k, H_{\mathsf{int}})$ as the infimum of $E_{k,H_{\mathsf{int}}}$ over all pairs $(u, \mathbf{a}) \in \mathcal{A}$.

For $u = 0$, $\mathbf{a} = 0$ and $H_{\mathsf{int}} = H_{\mathsf{ext}}$ one obtains the energy $E_{\mathcal{N}} = 1/4$, which is the energy of the so called *normal state*. In the limiting case $H_{\mathsf{int}} = 0$, one obtains (see [7] or [8]) the energy $E_{\mathcal{P}} = H_{\mathsf{ext}}^2/2$, which is the energy of the *pure state*. This leads us to introduce three sets in $\mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\begin{array}{rcl} \mathcal{N} & = & \{(k, H_{\mathsf{ext}}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ s.t. } \mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{N}}\} \ , \\ \mathcal{P} & = & \{(k, H_{\mathsf{ext}}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ s.t. } \mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{P}}\} \ , \\ \mathcal{M} & = & \{(k, H_{\mathsf{ext}}) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ s.t. } \mathcal{E}_{k,H_{\mathsf{ext}}} < \inf(E_{\mathcal{P}}, E_{\mathcal{N}})\} \ . \end{array}$$

The set $\mathcal{M}$ is the complementary of $\mathcal{P} \cup \mathcal{N}$ in $\mathbb{R}_+^* \times \mathbb{R}_+^*$; if $(k, H_{\mathsf{ext}}) \in \mathcal{M}$, then the superconductor is said to be in a *mixed state*.

Using this simple modelization we were able (see [7] and [8]) to prove following *monotonicity theorem*.

Theorem 1 *(i) If $(k, H_{\mathsf{ext}}) \in \mathcal{P}$, $k' \leq k$ and $H'_{ext} \leq H_{\mathsf{ext}}$ then $(k', H'_{ext}) \in \mathcal{P}$.*

(ii) If $(k, H_{\mathsf{ext}}) \in \mathcal{N}$, $k' \geq k$ and $H'_{ext} \geq H_{\mathsf{ext}}$ then $(k', \frac{k'}{k} H'_{ext}) \in \mathcal{N}$.

The existence of such a Theorem is possible only because the system is invariant by homotheties (see, for example, [9] for the case of a superconductor restricted to a domain $\mathcal{D}$ of $\mathbb{R}^2$).

From this theorem we derived the existence of two functions $k \mapsto H_{c1}(k)$ and $k \mapsto H_{c2}(k)$ such that

$$\begin{array}{rcl} \mathcal{N} & = & \{(k, H_{\mathsf{ext}}), \text{ s.t. } H_{\mathsf{ext}} \geq H_{c2}(k)\}, \\ \mathcal{P} & = & \{(k, H_{\mathsf{ext}}), \text{ s.t. } H_{\mathsf{ext}} \leq H_{c1}(k)\}, \\ \mathcal{M} & = & \{(k, H_{\mathsf{ext}}), \text{ s.t. } H_{c1}(k) < H_{\mathsf{ext}} < H_{c2}(k)\} \ . \end{array}$$

Using this modelization, we obtained in [8] the qualitative form of the phase diagram depicted in Fig. 1, which is described in Section 4.

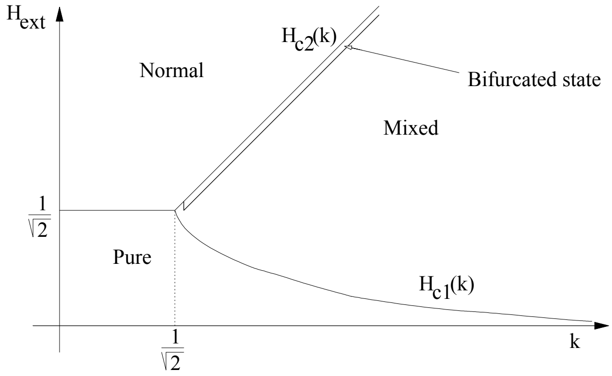

Figure 1: Phase diagram in Abrikosov modelization

This phase diagram is made of three curves:

(i) (*boundary normal-pure*) $H_{\mathsf{ext}} = H_{c1}(k) = H_{c2}(k) = 1/\sqrt(2)$ with $k \leq 1/\sqrt(2)$,

(ii) (*boundary normal-mixed*) $H_{\mathsf{ext}} = H_{c2}(k) = k$ with $k \geq 1/\sqrt(2)$,

(iii) (*boundary pure-mixed*) $H_{\mathsf{ext}} = H_{c1}(k)$ with $k \geq 1/\sqrt(2)$.

The exact expression of curve (iii) is unknown. These three curves meet at the triple point $k = H_{\mathsf{ext}} = 1/\sqrt(2)$. A key point of the proof is that the case $k = 1/\sqrt(2)$ is exactly solvable thanks to the Bochner-Kodaira-Nakano formula explained in Section 3. Using a more advanced analysis of the case $k = 1/\sqrt(2)$ in Section 5, we prove in Section 6 the following Theorem:

Theorem 2 *(i) There exist $\delta > 0$ and $S > 0$ such that for all h in $[0, \delta]$, we have*

$$-h \leq H_{c1}(\frac{1}{\sqrt{2}} + h) - \frac{1}{\sqrt{2}} \leq -Sh \ .$$

(ii) The critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = 1/\sqrt(2)$.

2 The change of variable

In this Section, we recall the original formulation of the problem by V. Ginzburg and L. Landau in [1] and how it is related to our formulation. They proposed the following expression for the density of energy in superconductors

$$\frac{1}{2}\|ik^{-1}\boldsymbol{\nabla}\phi + \mathbf{A}\phi\|^2 + \frac{1}{4}(1-|\phi|^2)^2 + \frac{1}{2}(\text{curl } \mathbf{A} - H_{\mathsf{ext}})^2.$$

This expression belongs to $L^1_{\mathsf{loc}}(\mathbb{R}^3)$ if $(\phi, \mathbf{A})$ is in the Sobolev space

$$H^1_{\mathsf{loc}}(\mathbb{R}^3;\mathbb{C}) \times H^1_{\mathsf{loc}}(\mathbb{R}^3;\mathbb{R}^3).$$

It is invariant under the gauge transformation $(\phi', \mathbf{A}') = (\phi e^{ikg}, \mathbf{A} + \boldsymbol{\nabla} g)$ with $g \in H^2_{\mathsf{loc}}(\mathbb{R}^2)$; this property is shared by other physically significant quantities such as the density of superconducting electrons $|\phi|^2$, the magnetic field $\mathbf{B} = \text{curl } \mathbf{A}$ and the current vector of superconducting electrons

$$Re[\overline{\phi}(ik^{-1}\boldsymbol{\nabla}\phi + \mathbf{A}\phi)].$$

We assumed that the problem is invariant under translation along the z-axis. This means that we consider pairs $(\phi, \mathbf{A})$, which satisfy: for every $h \in \mathbb{R}$, the pair $(\phi, \mathbf{A})(x_1, x_2, x_3 + h)$ is gauge equivalent to the pair $(\phi, \mathbf{A})$.

In fact, as proved in [7] p. 17, we can assume that the pairs $(\phi, \mathbf{A})$ considered are independent of x_3 and satisfy $\mathbf{A}_{x_3} = 0$. So, we can reduce the problem to a 2-dimensional one.

Let us take $\mathcal{L}$ as a 2-dimensional lattice of $\mathbb{R}^2$ with fundamental domain Ω of area 1. We consider the dilated lattice: $\mathcal{L}_\lambda = \sqrt{\lambda}\mathcal{L}$ with fundamental domain $\Omega_\lambda = \sqrt{\lambda}\Omega$. Following Abrikosov, we choose λ in $\mathbb{R}_+$ and restrict the analysis to pairs $(\phi, \mathbf{A})$, which are gauge periodic with respect to $\mathcal{L}_\lambda$ [6]. This means that for all $v \in \mathcal{L}_\lambda$, there exists $g^v \in H^2_{\mathsf{loc}}(\mathbb{R}^2)$ such that

$$\phi(z+v) = e^{ikg^v(z)}\phi(z) \;\text{ and }\; \mathbf{A}(z+v) = \mathbf{A}(z) + \boldsymbol{\nabla} g^v(z) \;.$$

Consequently, all the considered physical quantities are $\mathcal{L}_\lambda$-periodic. We denote by $|\Omega_\lambda|$ the area of Ω_λ, which is actually equal to λ.

A classic consequence (see [7, 10]) of gauge periodicity is that there exists $d \in \mathbb{Z}$ that satisfies the relation

$$2\pi d = k \int_{\Omega_\lambda} \text{curl } \mathbf{A} \;.$$

We will then, according to Abrikosov, fix the quantization d per unit cell equal to 1.

The Ginzburg-Landau functional is obtained by integration of the local density over the fundamental domain Ω_λ and division by $|\Omega_\lambda|$. This gives:

$$F(\phi, \mathbf{A}) = \frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \frac{1}{2} \|ik^{-1} \boldsymbol{\nabla} \phi + \mathbf{A}\phi\|^2 + \frac{1}{4}(1 - |\phi|^2)^2 + \frac{1}{2}(\text{curl } \mathbf{A} - H_{\mathsf{ext}})^2,$$

which should be understood as a mean energy.

We denote by $H_{\mathsf{int}} = \frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \text{curl } \mathbf{A}$ the mean internal magnetic field induced by $\mathbf{A}$. The quantization relation is then rewritten as $2\pi = k\lambda H_{\mathsf{int}}$.

It is also a classical result (see [10, 11] or [7], p. 21-29) that we can associate to the pair $(\phi, \mathbf{A})$, another pair $(\phi', \mathbf{A}')$, with the same Ginzburg-Landau energy but satisfying the following relation

(i) $\mathbf{A}' = \frac{H_{\mathsf{int}}}{2\pi} \mathbf{A}_0 + \mathbf{P}$ with $\mathbf{P}$ $\mathcal{L}_\lambda$-periodic, div $\mathbf{P} = 0$, $\int_{\Omega_\lambda} \mathbf{P} = 0$,

(ii) $\phi'(z + v) = e^{ikg^v(z)} \phi'(z)$ with $g^v(x, y) = \frac{H_{\mathsf{int}}}{2}(v_x y - v_y x)$ for all $v \in \mathcal{L}_\lambda$.

This reduction is rather complicated and is performed by a suitable gauge transform and a translation in x, y. The relation relating $\phi'(z + v)$ to $\phi'(z)$ actually defines the sections of a complex line bundle over the torus $\mathbb{R}^2/\mathcal{L}$; the result obtained above is therefore a classification result.

With this expression one gets

$$\frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \frac{1}{2}(\text{curl } \mathbf{A} - H_{\mathsf{ext}})^2 = \frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \frac{1}{2}(\text{curl } \mathbf{P})^2 + \frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2.$$

This leads to the simple expression $F(\phi, \mathbf{A}) = F^{int}(\phi, \mathbf{P}) + \frac{1}{2}(H_{\mathsf{int}} - H_{\mathsf{ext}})^2$ with

$$F^{int}(\phi, \mathbf{P}) = \frac{1}{\lambda} \int_{\Omega_\lambda} \frac{1}{2} \|ik^{-1} \boldsymbol{\nabla} \phi + \mathbf{A}\phi\|^2 + \frac{1}{4}(1 - |\phi|^2)^2 + \frac{1}{2}(\text{curl } \mathbf{P})^2.$$

The functional F^{int} is called internal energy and depends only on H_{int}, k, ϕ and $\mathbf{P}$.

The quantities H_{int}, k and λ are related by the quantization relation $2\pi = k\lambda H_{\mathsf{int}}$, which makes the analysis of F^{int} cumbersome. So, we reduce the complexity of the computation by the following change of variables and functions:

$$\begin{cases} u(x) = \phi(x\sqrt{\frac{2\pi}{kH_{\mathsf{int}}}}) , \\ \mathbf{a}(x) = \sqrt{\frac{2\pi k}{H_{\mathsf{int}}}}[\mathbf{A} - \frac{H_{\mathsf{int}}}{2\pi}\mathbf{A}_0](x\sqrt{\frac{2\pi}{kH_{\mathsf{int}}}}) = \sqrt{\frac{2\pi k}{H_{\mathsf{int}}}}\mathbf{A}(x\sqrt{\frac{2\pi}{kH_{\mathsf{int}}}}) - \mathbf{A}_0(x) . \end{cases}$$

We then obtain the formulation given in the introduction since the pair $(u, \mathbf{a})$ so defined belongs to $\mathcal{A}$ and verifies $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) = F^{int}(\phi, \mathbf{P})$.

3 The functional $E_{k,H_{\mathsf{int}}}$

Let us now analyze the functional $E_{k,H_{\mathsf{int}}}$ by assuming here that k and H_{int} are fixed.

$E_{k,H_{\mathsf{int}}}$ is defined over $\mathcal{A}$ since $(u, \mathbf{a})$ of class H^1 guarantees local integrability of the density, while the compactness of the torus $\mathbb{R}^2/\mathcal{L}$ guarantees its integrability.

In fact, the variational theory of the functional $E_{k,H_{\mathsf{int}}}$ is easy (see [7]) since the torus $\mathbb{R}^2/\mathcal{L}$ is compact and the non-linear partial differential equations obtained for the critical points are elliptic; the vector bundle adds only technical difficulties (see [12]). More precisely one can prove successively that:

1. *Coerciveness*: for every $C \in \mathbb{R}$ there is a $C' > 0$ such that $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) < C$ implies $\|u\|_{H^1} + \|\mathbf{a}\|_{H^1} \leq C'$.

2. *Lower semicontinuity*: If $(u_n, \mathbf{a}_n) \in \mathcal{A}$ converges weakly to $(u, \mathbf{a}) \in \mathcal{A}$, then $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) \leq \underline{\lim}_n E_{k,H_{\mathsf{int}}}(u_n, \mathbf{a}_n)$.

3. *Minimum*: The functional $E_{k,H_{\mathsf{int}}}$ attains its minimum on at least one pair $(u, \mathbf{a}) \in \mathcal{A}$.

4. *Ginzburg-Landau equations*: The minimizing pairs satisfy the following equation

$$\left\{ \begin{array}{rcl} \mu[i\boldsymbol{\nabla} + \mathbf{A}_0 + \mathbf{a}]^2 u &=& (1 - |u|^2)u \\ \Delta \mathbf{a} &=& \frac{1}{k^2} \operatorname{Re}[\overline{u}(i\boldsymbol{\nabla} u + (\mathbf{A}_0 + \mathbf{a})u)] \end{array} \right.$$

5. *Regularity*: The pairs $(u, \mathbf{a}) \in \mathcal{A}$ verifying the Ginzburg-Landau equations are in fact of class C^∞.

6. *Maximum principle*: The pairs $(u, \mathbf{a}) \in \mathcal{A}$ verifying the Ginzburg-Landau equations satisfy $|u| \leq 1$.

We now explain the Bochner-Kodaira-Nakano formula for the functional $E_{k,H_{\mathsf{int}}}$ (see [13 - 15] for related formulas and results). This classical formula is also called Bogomol'nyi formula, Weitzenbock formula, Lichnerowicz formula (see [16]) according to different scientific schools.

We set $\mathbf{C} = \mathbf{A}_0 + \mathbf{a}$; we get curl $\mathbf{C} = 2\pi +$ curl $\mathbf{a}$ and define

$$A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = \int_\Omega \frac{\mu}{2}|D_+ u|^2 + \frac{1}{4}|\mu \,\mathrm{curl}\, \mathbf{C} - (1 - |u|^2)|^2,$$

where $\mu = \frac{H_{\mathsf{int}}}{2\pi k}$ and $D_+ = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + C_y - iC_x$.

Theorem 3 *(*Bochner-Kodaira-Nakano*)*
For all $(u, \mathbf{a}) \in \mathcal{A}$*, we have :*

$$E_{\frac{1}{\sqrt{2}}, H_{\mathsf{int}}}(u, \mathbf{a}) = \mu\pi - (\mu\pi)^2 + A_{+,H_{\mathsf{int}}}(u, \mathbf{a}).$$

Proof. We perform computations with smooth functions and then extend by density. After expansion, simplification and regrouping one obtains

$$\begin{aligned} \{A_{+,H_{\mathsf{int}}} - E_{\frac{1}{\sqrt{2}}, H_{\mathsf{int}}}\}(u, \mathbf{a}) &= \tfrac{1}{2}\int_\Omega \mathrm{div}\ \mathbf{W} - \mu \,\mathrm{curl}\ \mathbf{C} \\ &+ \tfrac{\mu^2}{4}\int_\Omega |\,\mathrm{curl}\ \mathbf{C}|^2 - |\,\mathrm{curl}\ \mathbf{a}|^2 \end{aligned}$$

with

$$\mathbf{W} = \begin{pmatrix} \overline{u}(i\frac{\partial u}{\partial y} + C_y u) \\ -\overline{u}(i\frac{\partial u}{\partial x} + C_x u) \end{pmatrix}.$$

The vector field $\mathbf{W}$ being $\mathcal{L}$-periodic, the integral of its divergence over Ω is 0. The formula is then obtained by replacing curl $\mathbf{C}$ by $2\pi +$ curl $\mathbf{a}$ and using $\int_\Omega \mathrm{curl}\ \mathbf{a} = 0$.

The *magnetic Schrödinger* operator is defined as $H = [i\mathbf{\nabla} + \mathbf{A}_0]^2$; its spectrum, called *Landau levels*, is recalled in next theorem.

Theorem 4 *(i) The operator* H *admits a self-adjoint extension over* $L^2(E_1)$*, also denoted by* H*, whose domain is* $H^2(E_1)$*.*

(ii) It can be expressed as $H = L_+^* L_+ + 2\pi$ *with* $[L_+, L_+^*] = 4\pi$ *and* $L_+ = 2\partial_{\overline{z}} + \pi z$*.*

(iii) Its spectrum is discrete, $sp(H) = 2\pi + 4\pi\mathbb{N}$*, and every eigenvalue is simple.*

(iv) The eigenvector u_0 *associated to* $\lambda = 2\pi$ *satisfies* $L_+(u_0) = 0$ *and has a unique simple zero in* Ω *denoted by* z_0*.*

Proof. (i) and the discreteness of the spectrum follow from the fact that H is an elliptic pseudo-differential operator of order 2 defined over the vector bundle of a compact manifold (see [12]).

The formulae $H = L_+^* L_+ + 2\pi$ and $[L_+, L_+^*] = 4\pi$ are proved by first computing with smooth functions and then extending by density.

If we proved that the equation $L_+(u) = 0$ has a unique solution u_0 up to a scalar, then by the harmonic oscillator formalism we would get (iii).

In fact, if one writes, $u_0(z) = e^{-|z|^2 \frac{\pi}{2}} s(z)$, then $s(z)$ is analytic. Furthermore, without loss of generality, we can assume that $\mathcal{L}$ is generated by the vectors $v_1 = (u, 0)$ and $v_2 = (w, r)$ with $ru = 1$. Then, after using gauge periodicity conditions, one finds the following expression for u_0:

$$u_0(x, y) = e^{i\pi xy} \sum_{n\in\mathbb{Z}} e^{-\pi(y+nu)^2} e^{\pi n^2 iwu + 2\pi nuix} .$$

This expression is a theta function; it is known that such functions have a unique simple zero in Ω (see [17]). Another method of proof is the use of Rouché Theorem as done in [7].

Theorem 5 *If $k \geq \frac{1}{\sqrt{2}}$ and $H_{\mathsf{int}} \geq k$, then $m_E(k, H_{\mathsf{int}}) = \frac{1}{4}$. Furthermore, the minimum is met only by the pair* $(0, 0)$.

Proof. We use following expansion of the functional $E_{k,H_{\mathsf{int}}}$:

$$\begin{array}{rcl} E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) & \geq & E_{\frac{1}{\sqrt{2}},H_{\mathsf{int}}}(u, \mathbf{a}) \\ & \geq & (\mu\pi) - (\mu\pi)^2 + \int_\Omega \frac{\mu}{2}|D_+u|^2 + \frac{1}{4}|2\mu\pi - 1 + \mu \,\mathrm{curl}\; \mathbf{a} + |u|^2|^2 \\ & \geq & (\mu\pi) - (\mu\pi)^2 + \frac{(2\mu\pi-1)^2}{4} + \frac{1}{4}\int_\Omega 2(2\mu\pi - 1)(\mu \,\mathrm{curl}\; \mathbf{a} + |u|^2) \\ & & +\frac{1}{4}\int_\Omega |\mu \,\mathrm{curl}\; \mathbf{a} + |u|^2|^2 \\ & \geq & \frac{1}{4} + \frac{2\mu\pi-1}{2}\int_\Omega |u|^2 . \end{array}$$

Then using the hypothesis $2\mu\pi - 1 = H_{\mathsf{int}}/k - 1 \geq 0$, we get $m_E(k, H_{\mathsf{int}}) \geq 1/4$ positivity in terms in the above equation.

Now assume that $E_{k,H_{\mathsf{int}}}(u, \mathbf{a}) = 1/4$; in fact, the last computation gives us the following equalities:

$$\left\{ \begin{array}{rclrcl} 0 & = & (2\mu\pi - 1)\int_\Omega |u|^2, & 0 & = & \int_\Omega |\,\mathrm{curl}\; \mathbf{a} + |u|^2|^2, \\ 0 & = & (k^2 - \frac{1}{2})\int_\Omega |\,\mathrm{curl}\; \mathbf{a}|^2, & 0 & = & \int_\Omega |D_+u|^2. \end{array} \right.$$

The second equality gives us curl $\mathbf{a} + |u|^2 = 0$, which integrated over Ω yields

$$\int_\Omega |u|^2 = -\int_\Omega \mathrm{curl}\; \mathbf{a} = 0$$

and then $u = 0$.

Now, using the equation div $\mathbf{a} = 0$, one obtains the equality $\mathrm{curl}^* \,\mathrm{curl}\; \mathbf{a} = \Delta\, \mathbf{a} = 0$. The potential vector $\mathbf{a}$ is $\mathcal{L}$ periodic; so, it has to be constant. Now, the property $\int_\Omega \mathbf{a} = 0$ yields $\mathbf{a} = 0$.

4 The phase diagram

Let us first consider the special case when $k = H_{\mathsf{ext}} = 1/\sqrt{2}$. We have the following Lemma:

Lemma 6 *One has*
(i) $E_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}(H_{\mathsf{int}}, u, \mathbf{a}) = \frac{1}{4} + A_{+,H_{\mathsf{int}}}(u, \mathbf{a})$,
(ii) $\mathcal{E}_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}} = \frac{1}{4}$,
(iii) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in \mathcal{P} \cap \mathcal{N}$.

Proof. (i) is in fact a rewriting of the Bochner-Kodaira-Nakano formula; it yields (ii) by positivity of $A_{+,H_{\mathsf{int}}}$, while (iii) is obtained by remarking that $E_{\mathcal{N}} = \frac{1}{4} = \frac{1}{2}(\frac{1}{\sqrt{2}})^2 = E_{\mathcal{P}}$.

Theorem 7 *(Type I superconductors) If $k \leq \frac{1}{\sqrt{2}}$, then:*

(i) If $H_{\mathsf{ext}} \leq \frac{1}{\sqrt{2}}$, then $\mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{P}}$ and $(k, H_{\mathsf{ext}}) \in \mathcal{P}$,

(ii) If $H_{\mathsf{ext}} \geq \frac{1}{\sqrt{2}}$, then $\mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{N}}$ and $(k, H_{\mathsf{ext}}) \in \mathcal{N}$.

Proof. Lemma 6 combined with Theorem 1.(i) gives the result in the case $H_{\mathsf{ext}} \leq \frac{1}{\sqrt{2}}$.

In particular, if $k \leq \frac{1}{\sqrt{2}}$ we have $(k, \frac{1}{\sqrt{2}}) \in \mathcal{P}$ and so, $\mathcal{E}_{k,\frac{1}{\sqrt{2}}} = E_{\mathcal{P}} = \frac{1}{2}(\frac{1}{\sqrt{2}})^2 = \frac{1}{4} = E_{\mathcal{N}}$; therefore Theorem 1.(ii) gives the conclusion in case $H_{\mathsf{ext}} \geq \frac{1}{\sqrt{2}}$.

Theorem 8 *(Type II superconductors) If $H_{\mathsf{ext}} \geq k \geq \frac{1}{\sqrt{2}}$, then:*

(i) If $H_{\mathsf{ext}} \geq k$, then $\mathcal{E}_{k,H_{\mathsf{ext}}} = E_{\mathcal{N}}$ and $(k, H_{\mathsf{ext}}) \in \mathcal{N}$,

(ii) If $H_{\mathsf{ext}} < k$, then $(k, H_{\mathsf{ext}}) \notin \mathcal{N}$.

Proof. Lemma 6 combined with Theorem 1.(ii) gives (i).

By setting $H_{\mathsf{int}} = H_{\mathsf{ext}}$, $u = \alpha u_0$, $\mathbf{a} = 0$ and doing a development of order 2 around the pair $(0, 0)$, one obtains

$$E_{k,H_{\mathsf{ext}}}(H_{\mathsf{ext}}, \alpha u_0, 0) = E_{k,H_{\mathsf{ext}}}(\alpha u_0, 0) = \frac{1}{4} + \frac{1}{2}(\frac{H_{\mathsf{ext}}}{k} - 1)\alpha^2 + o(\alpha^2) .$$

Since $k > H_{\mathsf{ext}} = H_{\mathsf{int}}$, one obtains for α small $E_{k,H_{\mathsf{ext}}}(H_{\mathsf{ext}}, \alpha u_0, 0) < \frac{1}{4}$; so, the energy will be lower than 1/4, i.e. $(k, H_{\mathsf{ext}}) \notin \mathcal{N}$.

5 Analysis of the case $k = 1/\sqrt{2}$

In this section we will find all pairs $(u, \mathbf{a})$ verifying $A_{+,H_{\mathsf{int}}}(u, \mathbf{a})$, thus getting the value of $m_E(1/\sqrt{2}, H_{\mathsf{int}})$. A similar study is done in [18] for a rectangular problem. In the book [14], the case considered is of u defined over $\mathbb{R}^2$, while in paper [19] the problem is considered over a Riemann surface. Also, in [14] it is proved that all critical points of the Ginzburg-Landau functional are solution of the Bogomol'nyi equations, but their proof does not apply to our case.

Papers [20, 21, 15] are devoted to the existence theorem concerning the Kazdan-Warner equation. They get as a byproduct existence Theorems for the self-dual equations.

Theorem 9 *(Kazdan-Warner, see [20]). If h is a positive function, $h \neq 0$, and $C^\infty(\mathbb{R}^2/\mathcal{L})$. If $A > 0$ then the equation*

$$-\Delta f + e^f h = A$$

has a unique solution f in $C^\infty(\mathbb{R}^2/\mathcal{L})$.

We define

$$\begin{cases} u_{H_{\mathsf{int}}} &= u_0 e^{f_{H_{\mathsf{int}}}} \\ \mathbf{a}_{\mathbf{H}_{\mathsf{int}}} &= \left(\frac{\partial f_{H_{\mathsf{int}}}}{\partial y}, -\frac{\partial f_{H_{\mathsf{int}}}}{\partial x}\right), \end{cases}$$

with $f_{H_{\mathsf{int}}}$ being the unique solution of $1 - 2\mu\pi = |u_0|^2 e^{2f} - \mu\Delta f$ and $\mu = \frac{H_{\mathsf{int}}}{\pi\sqrt{2}}$.

Let us introduce first the following family of sections of E_1:

$$u_h(x, y) = e^{i\pi(h_y x - h_x y)} u_0(z - h) \ .$$

Recall that z_0 is the zero of u_0 in $\mathbb{R}^2/\mathcal{L}$; the section u_h verifies the following easy properties

$$\begin{cases} u_h \in C^\infty(E_1), \qquad L_+(u_h) = 2\pi h u_h, \\ u_h(z) = 0 \text{ if and only if } z \in z_0 + h + \mathcal{L} \ . \end{cases}$$

Furthermore, for any $h \in \mathbb{R}^2$, $v \in \mathcal{L}$, there exists $\alpha \in \mathbb{R}$ such that

$$u_{h+v}(z) = e^{i\alpha} e^{2i\pi(v_y x - v_x y)} u_h(z) \ .$$

Theorem 10 *We assume* $H_{\mathsf{int}} \le \frac{1}{\sqrt{2}}$.

(i) If $(u, \mathbf{a}) \in \mathcal{A}$ *satisfies* $A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = 0$*, then there exists* $c \in \mathbb{R}$ *such that* $(u, \mathbf{a}) = (e^{ic} u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$.

(ii) The pair $(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$ *satisfies to*

$$\begin{cases} \int_\Omega (1 - |u_{H_{\mathsf{int}}}|^2)^2 = \mu^2[(2\pi)^2 + \int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2] \\ \int_\Omega \frac{\mu}{2} \| i\boldsymbol{\nabla} u_{H_{\mathsf{int}}} + (\mathbf{A}_0 + \mathbf{a}) u_{H_{\mathsf{int}}} \|^2 + \frac{\mu^2}{2} |\operatorname{curl} \mathbf{a}_{H_{\mathsf{int}}}|^2 = (\mu\pi) - 2(\mu\pi)^2 \ . \end{cases}$$

Proof. Let $(u, \mathbf{a}) \in \mathcal{A}$ be a pair satisfying $A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = 0$, it then verifies the following Bogomol'nyi equations

$$D_+ u = L_+ u + (a_y - i a_x) u = 0 \ \text{ and } \ 2\mu\pi + \mu \operatorname{curl} \mathbf{a} = 1 - |u|^2$$

and, by Theorem 3, minimizes the functional $E_{\frac{1}{\sqrt{2}}, H_{\mathsf{int}}}$. Therefore, as shown in Section 3, it satisfies the Ginzburg-Landau equations and hence, it is C^∞.

Since the vector bundle E_1 is non trivial the section u possesses at least one zero in $\mathbb{R}^2/\mathcal{L}$, which we write as $z_h = z_0 + h$.

The zero-set of the function u defined on $\mathbb{R}^2$ contains $z_h + \mathcal{L}$, while the zero-set of u_h is exactly $z_h + \mathcal{L}$; so, one defines on $\mathbb{R}^2 - (z_h + \mathcal{L})$ the function

$$f = \frac{u}{u_h} \ .$$

Since both u and u_h are sections of the vector bundle E_1, the function f is $\mathcal{L}$-periodic. The equation $D_+ u = 0$ is rewritten on $\mathbb{R}^2 - (z_h + \mathcal{L})$ as:

$$0 = 2(\partial_{\overline{z}} f) u_h + f D_+ u_h = 2(\partial_{\overline{z}} f) u_h + [2\pi h f + (a_y - i a_x) f] u_h \ ,$$

Since u_h is not zero on $\mathbb{R}^2 - (z_h + \mathcal{L})$ we obtain:

$$\partial_{\overline{z}} f = f w \ \text{ with } \ w = \frac{1}{2}[(-a_y - 2\pi h_x) + i(a_x - 2\pi h_y)] \ .$$

Note that the function w is defined on $\mathbb{R}^2$, also it is C^∞ and $\mathcal{L}$-periodic.

We now want to extend f to $\mathbb{R}^2$: it is a classical result of complex analysis that the equation $\partial_{\overline{z}} k = w$ has a C^∞ solution k on $\mathbb{R}^2$.

The function $g = f e^{-k}$ is defined on $\mathbb{R}^2 - (z_h + \mathcal{L})$, satisfies $\partial_{\overline{z}} g = 0$ and is therefore analytic. If $m \in z_h + \mathcal{L}$ then $u = O(z - m)$, since u is C^∞. The complex m is a simple zero of u_h, consequently $u_h^{-1} = O(|z - m|^{-1})$ and $f = O(1)$ at m.

The function g stays bounded around m and is analytic outside m. By a classical result of complex analysis, we get that g can be extended to m in a complex analytic function. The function g is extended to $\mathbb{C}$ and therefore, f too.

The function g is analytic. Therefore its zero set is discrete. There exists a translation Ω' of Ω such that the boundary $\partial\Omega'$ of Ω' does not meet any zero of g.

By the Rouché theorem, the number n of zeroes of g in Ω' is equal to :

$$n = \frac{1}{2\pi i}\int_{\partial\Omega'} \frac{\partial_z g}{g} dz = \frac{1}{2\pi i}\int_{\partial\Omega'} \frac{\partial_z f}{f} dz - \frac{1}{2\pi i}\int_{\partial\Omega'} \partial_z k dz = \frac{-1}{2\pi i}\int_{\partial\Omega'} \partial_z k dz.$$

The integral of $\frac{\partial_z f}{f}$ over $\partial\Omega'$ is zero, since f is $\mathcal{L}$-periodic.

Now using Stokes theorem, we get :

$$\begin{aligned} n &= \tfrac{-1}{2\pi i}\textstyle\int_{\partial\Omega'} \partial_z k dz = \tfrac{-1}{2\pi i}\int_{\Omega'} d(\partial_z k dz) = \tfrac{-1}{2\pi i}\int_{\Omega'} \partial_{\overline{z}}\partial_z k d\overline{z} \wedge dz \\ &= \tfrac{-1}{2\pi i}\textstyle\int_{\Omega'} \partial_z\partial_{\overline{z}} k d\overline{z} \wedge dz = \tfrac{-1}{2\pi i}\int_{\Omega'} \partial_z w d\overline{z} \wedge dz. \end{aligned}$$

The function w is $\mathcal{L}$-periodic. Consequently the function $\partial_z w$ is a $\mathcal{L}$-periodic function, which has integral zero over Ω'; so, $n = 0$.

Since $f = ge^k$, the function f has no zero over $\mathbb{R}^2$. Since $\mathbb{R}^2$ is simply connected there exists a complex valued C^∞ function ψ such that $f = e^\psi$.

The function ψ is not $\mathcal{L}$-periodic, but since the function f is $\mathcal{L}$-periodic and C^∞ there exist two integers n_1, n_2 such that

$$\psi(z + v_1) = \psi(z) + 2\pi i n_1 \quad \text{and} \quad \psi(z + v_2) = \psi(z) + 2\pi i n_2 \ .$$

We pose $v' = n_1 v_2 - n_2 v_1$, the function $\psi_2(z) = \psi(z) - 2\pi i \det(z, v')$ is $\mathcal{L}$-periodic, and we have :

$$u(z) = f(z)u_h(z) = e^{\psi_2(z)+2\pi i[xv'_y - yv'_x]}u_h(z) = e^{\psi_2(z)-i\alpha}u_{h+v'}(z)$$

with $\alpha \in \mathbb{R}$. We set $h_3 = h + v'$, $\psi_3 = \psi_2 - i\alpha$, and we rewrite u as:

$$u(z) = e^{\psi_3(z)}u_{h_3}(z)$$

with ψ_3 a $\mathcal{L}$-periodic C^∞ function. The Bogomol'nyi equations are rewritten as

$$\left\{ \begin{array}{rcl} \frac{\partial\psi_3}{\partial\overline{z}} &=& \frac{1}{2}[(-a_y - \pi h_{3,x}) + i(a_x - \pi h_{3,y})] \ , \\ 0 &=& 2\mu\pi - 1 + |u_{h_3}|^2 e^{2\,\mathrm{Re}\,\psi_3} + \mu \,\mathrm{curl}\,\mathbf{a}. \end{array} \right.$$

The real and imaginary parts of the first equation give us the expression of the potential vector:

$$\begin{cases} a_x &= \pi h_{3,y} + \dfrac{\partial \operatorname{Re} \psi_3}{\partial y} + \dfrac{\partial \operatorname{Im} \psi_3}{\partial x} , \\ a_y &= -\pi h_{3,x} - \dfrac{\partial \operatorname{Re} \psi_3}{\partial x} + \dfrac{\partial \operatorname{Im} \psi_3}{\partial y} . \end{cases}$$

The equation div $\mathbf{a} = 0$ is then rewritten as Δ Im $\psi_3 = 0$. Thus Im ψ_3 is constant, since it is $\mathcal{L}$-periodic. We now write $\psi_3 = f + ic$ with f a real C^∞, $\mathcal{L}$-periodic function; so, one has

$$a_x = \pi h_{3,y} + \frac{\partial f}{\partial y} \quad \text{and} \quad a_y = -\pi h_{3,x} - \frac{\partial f}{\partial x} .$$

The functions $\mathbf{a}$, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ have zero integral over Ω. So, we have $h_3 = 0$ and the zero of u in Ω is z_0.

One then obtains curl $\mathbf{a} = -\Delta f$ and the following equation for f:

$$0 = 2\mu\pi - 1 + |u_0|^2 e^{2f} - \mu\Delta f .$$

So, one gets $f = f_{H_{\mathsf{int}}}$; now the above equation can be rewritten as:

$$-\mu \operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}} = 2\mu\pi - 1 + |u_{H_{\mathsf{int}}}|^2 .$$

It yields $\int_\Omega |u_{H_{\mathsf{int}}}|^2 = 1 - 2\mu\pi$ and $\int_\Omega (1 - |u_{H_{\mathsf{int}}}|^2)^2 = \mu^2[(2\pi)^2 + \int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2]$, the second equation of (ii) is then obtained by Theorem 3.

Corollary 11 . *For every positive H_{int} one has:*

$$m_E(\frac{1}{\sqrt{2}}, H_{\mathsf{int}}) = \begin{cases} \frac{H_{\mathsf{int}}}{\sqrt{2}} - (\frac{H_{\mathsf{int}}}{\sqrt{2}})^2 & \text{if} \quad H_{\mathsf{int}} \leq \frac{1}{\sqrt{2}} , \\ \frac{1}{4} & \text{if} \quad H_{\mathsf{int}} \geq \frac{1}{\sqrt{2}} . \end{cases}$$

Proof. By Theorem 3, one has the inequality $m_E(\frac{1}{\sqrt{2}}, H_{\mathsf{int}}) \geq \frac{H_{\mathsf{int}}}{\sqrt{2}} - (\frac{H_{\mathsf{int}}}{\sqrt{2}})^2$, since $A_{+,H_{\mathsf{int}}} \geq 0$. This lower bound is attained by the pair $(u_{H_{\mathsf{int}}}, \mathbf{a}_{H_{\mathsf{int}}})$.

Theorem 5 gives the result if $H_{\mathsf{int}} \geq \frac{1}{\sqrt{2}}$.

Remark 1 *It can be shown that the pair $(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$ depends continuously on H_{int} and vanish for $H_{\mathsf{int}} = \frac{1}{\sqrt{2}}$, i.e. it is a* bifurcated state *(see* [7]*).*

6 Local study

We define

$$\begin{array}{rcl} H_k(u,\mathbf{a}) & = & \frac{1}{4\pi k}\int_\Omega \|i\boldsymbol{\nabla} u + (\mathbf{A}_0+\mathbf{a})u\|^2 \\ & + & \sqrt{[\frac{1}{2}+\frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl}\mathbf{a}|^2][\int_\Omega (1-|u|^2)^2]}\ . \end{array}$$

Theorem 12 *If $k \geq \frac{1}{\sqrt{2}}$ then $H_{c1}(k) = \inf_{(u,\mathbf{a})\in\mathcal{A}} H_k(u,\mathbf{a})$. If this infimum is attained on a pair, say, $(u',\mathbf{a}')\in\mathcal{A}$, then one has*

$$E_{k,H_{c1}(k)}(H_{\mathsf{int}},u',\mathbf{a}') = \frac{H_{c1}^2(k)}{2} \quad \textit{with} \quad H_{\mathsf{int}} = \frac{1}{2}\sqrt{\frac{\int_\Omega (1-|u'|^2)^2}{\frac{1}{2}+\frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl}\mathbf{a}'|^2}}\ .$$

Proof. By Section 1, we have that $(k, H_{\mathsf{ext}})\in\mathcal{P}$ is equivalent to:

$$E_{k,H_{\mathsf{int}}}(u,\mathbf{a}) + \frac{1}{2}(H_{int}-H_{ext})^2 \geq \frac{H_{ext}^2}{2}\ ,$$

which after simplification is equivalent to

$$\left\{\begin{array}{l} H_{\mathsf{int}}[\frac{1}{2}+\frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl}\mathbf{a}|^2] + \frac{1}{4H_{\mathsf{int}}}\int_\Omega (1-|u|^2)^2 \\ +\frac{1}{4\pi k}\int_\Omega \|i\boldsymbol{\nabla} u + (\mathbf{A}_0+\mathbf{a})u\|^2 \geq H_{ext}. \end{array}\right.$$

The minimum over $H_{\mathsf{int}}>0$ of the above expression is attained for

$$H_{\mathsf{int}} = \frac{1}{2}\sqrt{\frac{\int_\Omega (1-|u|^2)^2}{\frac{1}{2}+\frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl}\mathbf{a}|^2}}$$

which yields the Theorem.

The above expression of $H_{c1}(k)$ allows us to obtain $H_{c1}(k) = O(\frac{\ln k}{k})$ (see [7]). From Theorem 7, one has $H_{c1}(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$.

Theorem 13 *The set of pairs $(u,\mathbf{a})\in\mathcal{A}$ verifying $H_{\frac{1}{\sqrt{2}}}(u,\mathbf{a}) = \frac{1}{\sqrt{2}}$ is*

$$(e^{ic}u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$$

with $c\in\mathbb{R}$ and $0 < H_{\mathsf{int}} \leq \frac{1}{\sqrt{2}}$.

Proof. If $(u, \mathbf{a}) \in \mathcal{A}$ satisfies $H_{\frac{1}{\sqrt{2}}}(u, \mathbf{a}) = \frac{1}{\sqrt{2}}$, then one has

$$E_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}(H_{\mathsf{int}}, u, \mathbf{a}) = \frac{1}{4} \quad \text{and} \quad H_{\mathsf{int}} = \frac{1}{2}\sqrt{\frac{\int_\Omega (1-|u|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl} \mathbf{a}|^2}} \; .$$

By Lemma 6.(i), first equation simplifies to $A_{+,H_{\mathsf{int}}}(u, \mathbf{a}) = 0$, and then using Theorem 10 to $(u, \mathbf{a}) = (e^{ic} u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$.

When the expression of $(u, \mathbf{a})$ is substituted into the second equation, one obtains

$$4H_{\mathsf{int}}^2 = \frac{\int_\Omega (1-|u_{H_{\mathsf{int}}}|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2}\int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2} \; .$$

By Theorem 10.(ii), this relation is always satisfied.

Theorem 14 *(i) There exist $\delta > 0$ and $S > 0$ such that for all h in $[0, \delta]$, we have*

$$-h \le H_{c1}(\frac{1}{\sqrt{2}} + h) - \frac{1}{\sqrt{2}} \le -Sh \; .$$

(ii) The critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = \frac{1}{\sqrt{2}}$.

Proof. The expression of $H_{c1}(k)$ obtained in Theorem 12 gives us that the function $k \mapsto kH_{c_1}(k)$ is increasing; this yields the lower bound.

Now we will prove the upper bound by using the $(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})$ as quasi-modes. If $k = \frac{1}{\sqrt{2}} + h$ then we will have

$$H_k(u_{H_{\mathsf{int}}}, \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}) = \frac{1}{\sqrt{2}} - \frac{h}{2\pi}\int_\Omega \|i\boldsymbol{\nabla} u_{H_{\mathsf{int}}} + (\mathbf{A}_0 + \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})u_{H_{\mathsf{int}}}\|^2 + o(h) \; .$$

We get the following values of S using Bochner-Kodaira-Nakano

$$\begin{aligned} S &= \sup_{0<H_{\mathsf{int}}<\frac{1}{\sqrt{2}}} \tfrac{1}{2\pi}\textstyle\int_\Omega \|i\boldsymbol{\nabla} u_{H_{\mathsf{int}}} + (\mathbf{A}_0 + \mathbf{a}_{\mathbf{H}_{\mathsf{int}}})u_{H_{\mathsf{int}}}\|^2 \\ &= \sup_{0<H_{\mathsf{int}}<\frac{1}{\sqrt{2}}} [1 - \tfrac{H_{\mathsf{int}}}{\frac{1}{\sqrt{2}}} - \tfrac{H_{\mathsf{int}}}{2\pi^2\sqrt{2}}\textstyle\int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2] \end{aligned}$$

follows from Theorem 10(ii).

One may want now to know the exact value of S at $1/\sqrt{(2)}$. Using numerical simulations we obtain that the function

$$\chi(H_{\mathsf{int}}) = 1 - \sqrt{2}H_{\mathsf{int}} - \frac{H_{\mathsf{int}}}{2\pi^2\sqrt{2}}\int_\Omega |\operatorname{curl} \mathbf{a}_{\mathbf{H}_{\mathsf{int}}}|^2$$

is decreasing and has a limit of approximately 0.78 at $H_{\mathsf{int}} = 0$ for a square lattice.

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