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Thermoelastic vibrations with thermal relaxation time in multilayered media

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Abstract

A theoretical treatment is presented for interaction of thermoelastic waves with n-layered isotropic plates in the context of generalized theory of thermoelasticity, using the matrix transfer technique. The wave is supposed to be incident at an arbitrary angle, upon a plate consisting of an arbitrary number of different thermoelastic isotropic layers. Solutions for each layer are derived and expressed in terms of wave amplitudes. By eliminating these amplitudes the displacements, temperature, thermal stresses and temperature gradient are on one side of the layer are related to those of the other side. By satisfying appropriate conditions at interlayer interfaces a matrix is constructed which relates displacements, temperature, thermal stresses and temperature gradient on one side of the plate to those of the other side. Invoking appropriate boundary conditions on the plate outer boundaries, many important problems can be solved. The model developed will be of value in material characterization and others quantitative information on thermo-mechanical, strength related properties of advanced materials. The propagation of free waves on the plate and propagation of waves in periodic media consisting of a repetition of the plate are considered in this paper. A variety of numerical illustrations are included.

Keywords: Layered plates, generalized thermoelasticity, temperature gradient, thermal relaxation time.
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1 Introduction

Studies of the propagation of elastic waves in layered media (Ewing, Jardetsky, and Press [1], Brekhovskikh [2]) have long been of interest of researchers in the fields of geophysics, acoustics, and electromagnetic. Applications of these studies include such technologically important areas as earthquake prediction, underground fault mapping, oil and gas exploration, architectural noise reduction.

The theory to include the effect of temperature change, known as the theory of thermoelasticity, has also been well established. According to the theory, the temperature field is coupled with the elastic strain field. The classical theory of thermoelasticity predicts infinite speed of transportation, which contradicts the physical facts. Lord and Shulman [3], and Green and Lindsay [4] extended the coupled theory of thermoelasticity by introducing the thermal relaxation time in the constitutive equations. These new theories eliminate the paradox of infinite velocity of heat propagation are called generalized theories of thermoelasticity. Verma and Hasebe [5-7] have studied dynamic thermoelastic problems by considering hyperbolic heat conduction equation with thermal relaxation times. Verma [8] investigated thermoelastic vibrations in transversely isotropic plate with thermal relaxations and derived the dispersion equation.

Dynamic problems in layered thermoelastic media become more complicated than its counterpart in elasticity, because in thermoelasticity, solutions to both the heat conduction and thermoelasticity problems for all the layers are required. These solutions are also to satisfy the thermal and mechanical boundary and interface conditions. As a result, conventional procedure for thermal stresses analysis of a multilayered medium result in having to solve system of two simultaneous equations for a large number of unknown constants. Bufler [9], Bahar [10] proposed the transfer matrix method to study the isothermal elasticity problems in multilayered medium, and later extended to thermoelasticity by Bahar and Hetnarski [11]. Verma *et al.* [12] studied theoretical development for the multilayered plate with either all rigidly bonded interfaces or with one smooth interface and the remaining interfaces rigidly bonded is presented in the context of theories of generalized thermoelasticity for a plate consisting of an arbitrary number of isotropic layers. Nayfeh *et al.* [13-15] have studied wave propagation problems in multilayered elastic media. Similar approaches using transfer matrix method are also discussed in the articles and books [17-23].

In the present paper, a summary of the theoretical development for the multilayered plate with rigidly bonded interfaces is presented in the context of theory of generalized thermoelasticity of Lord and Shulman, for a plate composed of an arbitrary number of isotropic layers. The wave is allowed to propagate along at an arbitrary angle θ measured from the normal to the interface. The plate's lower boundary responses extended into the plate via the matrix transfer technique yielding expression for the displacements, temperature, thermal stresses and temperature gradient in terms of wave amplitudes. Eliminate these wave amplitudes, displacements, temperature, thermal stresses and temperature gradient on one side of the layer to those of the other side. Solutions obtained are general and pertain to several special cases. Of these mention: dispersion characteristics for a multilayered thermoelastic plate consisting of an arbitrary number of layers; dispersion of infinite medium built from repetition of the multilayered thermoelastic plate (periodic media consisting of a periodic repetition of the thermoelastic plate); and slowness results for either homogeneous periodic media. Confidence in the approach and results are confined by comparisons with whatever is available from specialized solutions. A variety of numerical illustrations are included.

Numerical results reveal that velocity of quasi-longitudinal wave is greater than or equal to the velocity of quasi-**t**hermal waves, for any angle of incidence and also depend upon thermal relaxation time in thermoelastic layered media. It has also been found that quasi-thermal wave speed increases, and approaches to quasi-longitudinal wave speed with increase in angle of incidence. It has also been observed that the maximum value of quasi-longitudinal wave speed decreases as angle of incidence increases.

2 Formulation

Consider a plate consisting of an arbitrary number, n, of homogeneous thermoelastic isotropic layers rigidly bonded at their interfaces. The problem is to study the propagation of thermoelastic waves in the context of generalized theory of thermoelasticity. We shall assume that a plane wave propagates in the $x - z$ plane at an arbitrary angle θ measured from the normal to the interface. We shall use two sets of two-dimensional coordinate systems $(x,\, z)$. One system is global coordinate system, which has its origin at the bottom layer of the plate such that x denotes the propagation direction and z is the normal to the interfaces. Hence layered plane will then occupy the space $0 \leq \mathrm{z} \leq \mathrm{d}$ where d denotes the total thickness of the plate. The second system is local for each sub-layer of the plate. Since the plate is made of n layers, the mth layer will then have its local coordinate $x^{(m)}$ and

$z^{(m)}$ with local origin at bottom surface. Hence each layer occupy the space $0 \leq z^{(m)} \leq d^{(m)}$ where $d^{(m)}$ is its thickness. With this choice of co-ordinate system the equation of motion and heat conduction for each layer are (Lord and Shulman [3])

$$\mu[u_{,xx} + u_{,zz}) + (\lambda + \mu)(u_{,xx} + w_{,xz}) = \rho\ddot{u} + \gamma T_{,x} \tag{2.1}$$

$$\mu[w_{,xx} + w_{,zz}) + (\lambda + \mu)(u_{,xx} + w_{,xz}) = \rho\ddot{w} + \gamma T_{,x} \tag{2.2}$$

$$K[T_{,xx} + T_{,zz}] + \rho C_e(\dot{T} + \tau_0\ddot{T}) = \gamma T_0[(\dot{u}_{,x} + \dot{w}_{,z} + \tau_0(\ddot{u}_{,x} + \ddot{w}_{,z}) \tag{2.3}$$

where τ_0 is the thermal relaxation time, λ and μ are Lame's constants, $\gamma = (3\lambda+2\mu)\alpha_t$ is the thermoelastic coupling constant and α_t the coefficient of thermal expansion and all other symbols have their usual meanings as in Lord and Shulman [3]. The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation.

3 Analysis

For waves who's projected wave vector is along the x-axis, and for an angle of incidence θ, Eqs. (2.1)-(2.3) admit the formal solutions

$$(u, w, T) = (U_1, U_2, U_3)\exp[i\xi(x\sin\theta + \alpha z - ct)] \tag{3.1}$$

where ξ is the wave number, U_1, U_2 and U_3 are the constant amplitudes related to displacements and temperature T, c is the phase velocity ($c = \omega/\xi$), ω is the circular frequency, α is the ratio of the z and x-directions wave numbers. This choice of solutions leads to the coupled equations

$$M_{mn}(\alpha)U_n = 0 \quad (m, n = 1, 2, 3) \tag{3.2}$$

where the summation convention is implied, and

$$\begin{aligned}
&M_{11} = c_2\alpha^2 + \sin^2\theta - \zeta^2,\\
&M_{12} = c_3\sin\theta\alpha,\\
&M_{13} = \sin\theta,\\
&M_{21} = M_{12},\\
&M_{22} = c_2\sin^2\theta + \alpha^2 - \zeta^2,\\
&M_{23} = \alpha,\\
&M_{31} = \varepsilon_1\tau\omega^*\zeta^2\sin\theta,\\
&M_{32} = \varepsilon_1\tau\omega^*\zeta^2\alpha,\\
&M_{33} = (\sin^2\theta + \alpha^2) - \omega^*\zeta^2\tau,\\
&c_2 = \mu/(\lambda + 2\mu), \quad \zeta^2 = \rho c^2/(\lambda + 2\mu). \quad c_3 = 1 - c_2
\end{aligned} \tag{3.3}$$

$$\varepsilon_1 = \frac{T_0\gamma^2}{\rho C_e(\lambda+2\mu)}, \omega^* = C_e(\lambda+2\mu)/K, \quad \tau = (\tau_0 + i/\xi c), \tag{3.4}$$

The existence of nontrivial solution for U_1, U_2 and U_3 demands the vanishing of the determinant in Eqs. (3.2), and yields the polynomial equation

$$\alpha^6 + P_1\alpha^4 + P_2\alpha^2 + P_3 = 0, \tag{3.5}$$

where P_1, P_2 and P_3 are given in Appendix 1.

Solving (3.5) for the six roots of α and using superposition results in the following formal solution relating the displacements, temperature, thermal stresses and temperature gradient within a layer to its wave amplitudes.

$$\begin{bmatrix} u \\ w \\ T \\ \bar{\sigma}_{zz} \\ \bar{\sigma}_{xz} \\ \bar{T}' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{\alpha_1}{\sin\theta} & -\frac{\alpha_1}{\sin\theta} & \frac{\alpha_2}{\sin\theta} & -\frac{\alpha_2}{\sin\theta} & -\frac{\sin\theta}{\alpha_3} & \frac{\sin\theta}{\alpha_3} \\ S_1 & S_1 & S_2 & S_2 & 0 & 0 \\ D_1 & D_1 & D_1 & D_1 & D_3 & D_3 \\ D_4 & -D_4 & D_5 & -D_5 & D_6 & -D_6 \\ D_7 & -D_7 & D_8 & -D_8 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_1E_1 \\ B_2E_2 \\ B_3E_3 \\ B_4E_4 \\ B_5E_5 \\ B_6E_6 \end{bmatrix} E \tag{3.6}$$

Here α_1^2, α_2^2 and α_3^2 are roots of equation (3.5) and taking $\alpha_{j+1} = -\alpha_j$, $j = 1, 2, 3$

$$E_q = e^{i\xi\alpha_q z}, E = e^{i\xi(x\sin\theta - ct)}, \ q = 1, 2...6,$$

α_1^2, α_2^2 correspond to coupled longitudinal and thermal waves, whereas α_3^2 corresponds to transverse wave which is not affected by the temperature variations

$$\begin{aligned} &D_1 = c_2(\tfrac{c^2}{C_S^2} - 2\sin^2\theta), \quad D_3 = -2c_2\sin\theta, \\ &D_4 = 2c_2\alpha_1, D_5 = 2c_2\alpha_2, D_6 = \tfrac{c_2}{\alpha_3}[\tfrac{c^2}{C_S^2} - (1+\sin^2\theta)], \\ &D_7 = \alpha_1 S_1, \quad D_8 = \alpha_2 S_2, \quad C_S^2 = \tfrac{\mu}{\rho}, \quad q = 1, 2, \\ &\bar{\sigma}_{zz} = \tfrac{\sigma_{zz}}{i\xi}, \bar{\sigma}_{xz} = \tfrac{\sigma_{xz}}{i\xi}, \bar{T}' = \tfrac{T'}{i\xi}. \end{aligned} \tag{3.7}$$

The various parameters $\alpha_1, \alpha_2, \alpha_3, S_1, S_2, D_1$, etc. are specialized to the material of layer (m) under consideration. Specializing (3.6) to the upper and bottom surface of each layer we can relate, after lengthy algebraic reductions and manipulations, the displacements, temperature, stresses and temperature gradient of the upper to those of the bottom as

$$P_m^+ = A_m P_m^-, \quad m = 1, 2......n \tag{3.8}$$

where

$$P_m^{\pm} = \{[u, w, T, \sigma_{zz}, \sigma_{xz}, T']\}_m \tag{3.9}$$

defines the variables column specialized to the upper and lower surfaces of the layer, m , respectively, and which provide us

$$A_m = X_m R_m X_m^{-1} \tag{3.10}$$

where is X_m is 6×6 square matrix of (3.6) and R_m is 6×6 diagonal matrix whose entries are $E_q = \exp(i\xi\alpha_q d)$.The matrix $A_m$constitutes the transfer matrix for the layer m. By applying the above procedure to each layer and invoking the continuity relations on the upper and bottom of each layer to those of its neighbors we can finally relate the displacements, temperature, stresses and temperature gradient at the top (top of layer (n)) of the plate to those at the bottom of the plate (bottom of layer (1)) via the transfer matrix multiplication. We can finally relate the displacements, stresses, temperature and temperature gradient at the top of the layered plate, $z = d$ to those at its bottom, $z = 0$ via the transfer matrix multiplication

$$A = A_n A_{n-1} .... A_1, \tag{3.11}$$

which leads to

$$\begin{bmatrix} u^{(n)} \\ w^{(n)} \\ T^{(n)} \\ \bar{\sigma}_{zz}^{(n)} \\ \bar{\sigma}_{xz}^{(n)} \\ \bar{T'}^{(n)} \end{bmatrix}_{z=d} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ w^{(1)} \\ T^{(1)} \\ \bar{\sigma}_{zz}^{(1)} \\ \bar{\sigma}_{xz}^{(1)} \\ \bar{T'}^{(1)} \end{bmatrix}_{z=0}, \tag{3.12}$$

which provide us

$$P^+ = AP^- \tag{3.13}$$

where P^+ and P^- are the displacement, temperature, stress, and temperature gradient column vectors at the top $z = d$ and bottom $z = 0$ of the total plate, respectively.

4 Transfer matrix A

Since Eq. (3.14) holds for any number of layers n, then it holds for a single layer in particular, and thus A can be represented by A_m for $m = 1, 2....n$. Accordingly, we hypothesize that any general property of A_m is also a property of A. It can be verified that the matrix A_m possesses the properties

(a) $$\det(A_m) = 1. \quad (4.1)$$

As R_m is diagonal, its determinant is equal to the product of its diagonal entries, which by employing the properties of $\alpha_2 = -\alpha_1, \alpha_4 = -\alpha_3, \alpha_6 = -\alpha_5$ of the roots of Eq. (3.5) is seen to be unity.

(b) $$\det(A_m) = \det(R_m) \quad (4.2)$$

The matrices A_m and R_m are similar and hence have same eigen values say λ_q, $q = 1, 2, 3...6$. The eigenvalues of A_m are given by the diagonal elements of R_m. By inspection we see that if λ is an eigenvalue A_m so is $\frac{1}{\lambda}$. Thus if $\lambda_1, \lambda_3, \lambda_5$ are eigenvalues of A_m, so are

(c) $$\lambda_{j+1} = 1/\lambda_j, \quad j = 1, 3, 5. \quad (4.3)$$

The above results and the fact that the eigen values of A_m^{-1} are the inverse of the corresponding eigenvalues of A_m lead to the conclusion that A_m and A_m^{-1} have the same set of eigenvalues.

As a consequence of (a) to (c) and the definition (3.11) we have

(i) $$\det(A) = \det(A_n)\det(A_{n-1}.).......\det(A_1) = 1 \quad (4.4)$$

(ii) The eigenvalues of A^{-1} are equal to the eigenvalues of A.

Proof: Let us assume that the eigenvalues of A is σ. Then σ^{-1} is the eigenvalue of A^{-1}. By substituting from (3.9) and (3.11) and carrying the inverse of A, we get

$$\det(X_m R_m X_m^{-1}...X_2 R_2 X_2^{-1} X_1 R_1 X_1^{-1} - \sigma I) = 0 \quad (4.5)$$

$$\det(X_1 R_1^{-1} X_1^{-1} X_2 R_2^{-1} X_2^{-1}...X_m R_m^{-1} X_m^{-1} - \sigma^{-1} I) = 0. \quad (4.6)$$

Using the fact that the product of the two equal rank square matrices PQ and QP (although $PQ \neq QP$) have the same eigenvalues [11], by cyclic permutation, we can rewrite the relation (4.5) as

$$\det(X_1 R_1 X_1^{-1} X_2 R_2 X_2^{-1}...X_m R_m X_m^{-1} - \sigma I) = 0. \quad (4.7)$$

By inspection (4.6) and (4.7) implies that (4.7) can be obtained from (4.6) by just inverting the diagonal matrices D_k and the eigenvalues σ. Since the entries (eigenvalues) of R_k and D_k^{-1} are the same, therefore we conclude that the eigenvalues σ_q, $q = 1, 2, 3, ...6$ and σ_q^{-1} form the same set.

(iii) When all the layers are made up of the same material (degenerate case) but not necessarily have equal thickness; then X_k and the eight values α_q are the same for every k. Now substituting from (3.11) and (3.12)) and recognizing that $X_{i+1}^{-1} X_i = I$ for $i = 1, 2, ...n$ the global matrix collapses

$$A = X_1 R X_1^{-1} \tag{4.8}$$

where we have used the fact that $X_n = X_1$, and

$$R = R_n R_{n-1} ... R_1 \tag{4.9}$$

is a diagonal matrix whose entries are given by $e^{i\xi\alpha_q d}$, $q = 1, 2, 3...6$, and d is the total thickness. Thus we have shown that the global transfer matrix correctly reduces to the corresponding matrix of the single material plate when all layer properties are same.

(iv) If we expand the characteristic equation $\det(A - \sigma I) = 0$ and write in terms of both the eigen values σ_q and invariants I_q of A, $q = 1, 2, \ldots 6$ and compare the resulting expressions, we conclude the symmetric relations

$$I_5 = I_1, \quad I_4 = I_2, \quad I_6 = 1. \tag{4.10}$$

The results $I_6 = 1$ also confirm the fact that $\det(A) = 1$.

Equation (3.14) will now be used to presents solution for a variety of situations.

Firstly, we consider a single cell medium, namely a free n-layered plate. The characteristic equation for such a situation is obtained by invoking stress-free as well as thermally insulated upper and bottom surfaces in equation (3.14).

We obtain the characteristic equation as

$$\begin{vmatrix} A_{41} & A_{42} & A_{43} \\ A_{51} & A_{52} & A_{53} \\ A_{61} & A_{62} & A_{63} \end{vmatrix} = 0. \tag{4.11}$$

5 Classical Floquet periodically condition

A second important situation is that of a periodic medium consisting of a repetition of the unit cell (plate). Here we generalize the classical Floquet periodically condition to require

$$P^{+} = P^{-1} e^{i\xi d \cos\theta} \tag{5.1}$$

which is consistent with the formal solution (3.1). Combinations of (4.1) and (5.1) yields the characteristic equation

$$\det(A - I e^{i\xi d \cos\theta}) = 0. \tag{5.2}$$

Equation (5.2) after expansion can be written in terms of the invariants I_q of A, which after algebraic manipulation reduces to

$$\cos(3\xi d\cos\theta) - I_1 \cos(2\xi d\cos\theta) + I_2 \cos(\xi d\cos\theta) - I_3/2 = 0. \tag{5.3}$$

In terms of the individual entries A_{ij} of A the invariants are given in [16] by

$$(-1)^m I_m = \sum_{i_1<i_2<...<i_m} \Delta(i_1, i_2, ...i_m) \tag{5.4}$$

where $\Delta(i_1, i_2, ...i_m)$ is the m-th order determinant formed from the rows $i = i_1, i_2, ...i_m$ and columns $j = i_1, i_2, ...i_m$.

6 Special cases

(a) To show the extent of generality in the results, we now discuss the case in which all layers are the same. This is expected to undoubtedly result in a description of the behavior of single homogeneous isotropic thermoelastic materials as discussed earlier, the global transfer matrix for such a situation collapses to form given in (4.11). Using this matrix, together with the fact that for the case D and A are similar, dictates that the characteristic equation (5.3) admits the solution

$$\alpha_q^2 = \cos^2\theta, \quad q = 1, 2, ...6 \tag{6.1}$$

which also specializes the formal solution (3.1) to the one appropriate for the single homogeneous thermoelastic medium. With reference to equation (3.5) and for a fixed θ, result (5.3) admits three roots for phase velocity c corresponding to one quasi-longitudinal, one transverse and one quasi-thermal motion. Thus for a variable θ, equation (6.1) describes the variation of four phase velocities with the incident angle and hence constitute wave front curves. For this specialized single medium case, these curves will be independent of frequency, however.

For an example equation (3.5) gives

$$\alpha_1^2\alpha_2^2\alpha_3^2 = [(\frac{c^2}{C_L^2} - \sin^2\theta)(\frac{c^2}{C_T^2} - \sin^2\theta) - \varepsilon_1 \sin^2\theta \frac{c^2}{K_T^2}](\frac{c^2}{C_S^2} - \sin^2\theta) \quad (6.2)$$

where $C_L^2 = \dfrac{\lambda+2\mu}{\rho}$, $C_S^2 = \dfrac{\mu}{\rho}$, $C_T^2 = \dfrac{K^*}{\rho}$, $K^* = \dfrac{K}{C_e\tau}$, and $\varepsilon_1 = \dfrac{\gamma^2 T_0}{C_e(\lambda+2\mu)\rho}$, where C_L, C_T, K_T, and ε_1 are the purely longitudinal, shear, and thermal wave speeds and thermal coupling constant in the medium, respectively.

When the strain and thermal fields are uncoupled to each other i.e. $\varepsilon_1 = 0$, then equation (6.2) reduces to

$$\alpha_1^2 = (\frac{c^2}{C_L^2} - \sin^2\theta),\ \alpha_2^2 = (\frac{c^2}{C_T^2} - \sin^2\theta) \text{ and } \alpha_3^2 = (\frac{c^2}{C_S^2} - \sin^2\theta). \quad (6.3)$$

Thus, combination of equation (6.1) and (6.3) gives the roots $c = C_L$, $c = C_S$ and $c = C_T$ yielding three concentric spherical wave front curves as is expected. Clearly thermal wave front is influenced by thermal variations.

(b) *Layered media:* in this case, the situation is much more complicated due to the dependence of the phase velocities, not only on the individual layer properties, having thermoelastic parameters, but most importantly on the wave number. However, for a fixed frequency, we can construct wave front curves and hence by varying the frequency or wave number in a discrete manner, demonstrate a frequency-dependent "dispersive" character of wave front curves.

(c) *Classical case:* This case corresponds to the situation when the strain and temperature fields are not coupled with each other. In this case the thermo-mechanical coupling constant is identically zero. In this case

$$M_{31} = M_{32} = 0. \quad (6.4)$$

The equation (3.5) with the help of (6.4) reduces to

$$\{\alpha_1^2 + \sin^2\theta - \zeta^2\}\{c_2\alpha_2^2 + c_2\sin^2\theta - \zeta^2\}\{\alpha_3^2 + \sin^2\theta - \omega^*\tau\zeta^2\} = 0 \quad (6.5)$$

which gives

$$\alpha_1^2 = \zeta^2 - \sin^2\theta, \quad \alpha_2^2 = \omega^*\tau\zeta^2 - \sin^2\theta \text{ and } \alpha_3^2 = \frac{\zeta^2}{c_2} - \sin^2\theta \tag{6.6}$$

α_1^2, α_2^2 correspond to longitudinal and transverse wave respectively, and α_3^2 corresponds to the thermal wave, which is clearly influenced by the thermal relaxation time τ_0.

7 Discussion and numerical illustration

Equations (2.1)-(2.3) serve as a coupled system for the displacement and temperature fields. In (6.2), C_L, C_S represents the speeds of purely elastic dilatational and shear waves, and $C_T$represents the speed of purely thermal waves in the absence of coupling constant ε_1. When the strain and thermal fields are coupled to each other, the longitudinal and thermal waves are coupled, the distinction between them is somewhat artificial and the speed of two waves are being considered quasi-longitudinal wave speed c_{QL} and quasi-thermal c_{QT}.

The analytical results (4.11) and (5.3) may further be simplified, once the number of layers, their properties, and geometric stacking are specified, and then one can present numerical results in two categories. In the first, variations of phase velocity c with the angle of incidence θ for specified wavenumber, this is effectively a form of demonstration of the dependence of wave front (inverse of slowness) curves with frequency. Secondly, phase velocity dispersion curves can be plotted as function of the product of frequency and unit cell thickness, for specified angle of incidence θ. Conventional dispersion curves in the forms of variation of wave velocities with wave number can also be constructed using (4.11) and (5.3).

Sample examples, which demonstrate the dependence of such curves on θ, and thermal relaxation time are presented for a periodic media lay-up configurations, composed of aluminum alloy/carbon steel plates in LS theory. The physical data for aluminum alloy and carbon steel is given in Appendix 2. The material ordering for the plate is given by Al/C/Al/C, and for periodic media lay-up all of these plates have the same thickness d.

In Fig. 1 the dispersion curves in the forms of variations of wave velocities (normalized) with wave-number (normalized) are constructed using equation (5.3) for fixed values of angle of incidence θ, and thermal relaxation time. Close examination of these figures reveals several interesting features.

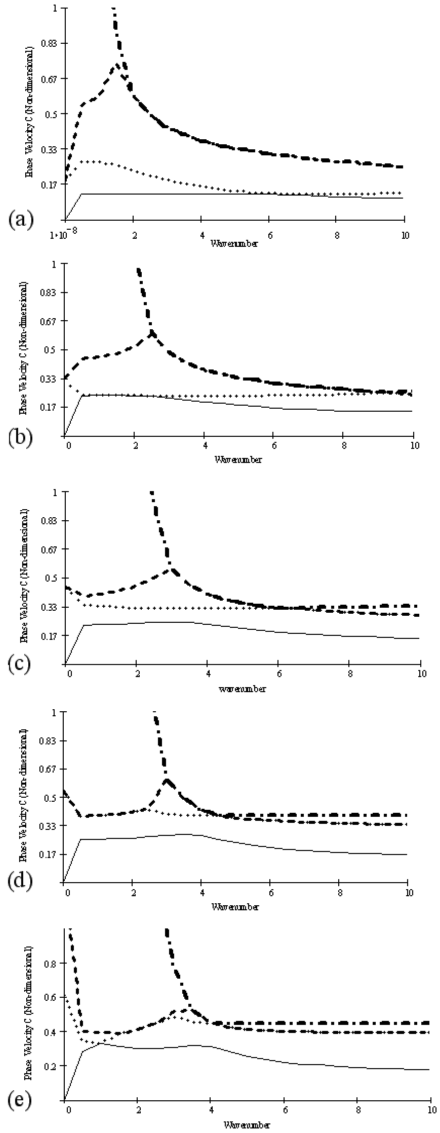

Figure 1: Variation of phase velocity C with wave-number ξ for angle of incidence θ (a) 15^0, (b) 30^0, (c) 45^0, (d) 60^0, and (e) 90^0. Modes: —— transverse mode, $\cdot\cdot\cdot$ quasi-thermal mode, - - - quasi-longitudinal mode, $-\cdot-\cdot$ higher mode.

At the zero wave-number limits, each figure displays three values of wave speeds corresponding to one quasi-longitudinal and one quasi-thermal and transverse. It is obvious that the largest value corresponds to the quasi-longitudinal mode. At relatively low values of the wave number, little change is seen to take place in these values. As ξ increases, other higher order modes appear; one of these seems to be associated with a rapid change in the slope of quasi-longitudinal mode. In Fig. 1, when angle of incidence increase from $\theta = 15^0$, to $\theta = 90^0$, quasi-longitudinal wave speed is found to be greater than or equal to thermal wave speed, i.e. ($c_{QL} \geq c_{QT}$).

In Fig. 1(a), when $\theta = 15^0$, thermal wave speed found very close to transverse wave speed near $\xi = 6\text{mm}^{-1}$, and then approaches to quasi-longitudinal mode when θ increases from $\theta = 15^0$, to $\theta = 90^0$. In Fig. 1(d), when $\theta = 60^0$, and $\xi < 2.7$, $c_{QL} > c_{QT}$, and in this case c_{QL} and c_{QT} found very close to each other, when $\xi > 4.5$, $c_{QL} > c_{QT}$. In Fig. 1(e), when $\theta = 90^0$, for $\xi < 4.2$, and $c_{QL} = c_{QT}$ when $\xi \geq 4.2$, and both values approaches to 0.3947315 (non-dimensional). It is observed that when $\theta > 45^0$, $c_{QL} \geq c_{QT}$, and when $\theta \leq 45^0$, $c_{QL} > c_{QT}$.

In fact, Figs. 1(a)-(e) shows a sharp variation in c_{QL}, c_{QT} from one layer to the next reflecting the change in materials properties through the interface and are influenced by angle of incidence, thermal relaxation time τ_0. Generally, available literature on layered media is restricted to the study of situations where the individual material layers are purely elastic, and hardly any problem has attempted in generalized thermoelasticity considering layered media. Furthermore, dispersion behavior of curves in Figs.1(a) - 1(e) is in agreement with [14].

In Fig. 2 we depict, for the selected values of wave number and thickness, wave front curves in the X-Y plane where$X = c\sin\theta$ and $Y = c\cos\theta$ lay-up periodic medium as a representative case. The curves demonstrate the inverse of the slowness curves as functions of wave number and hence display and demonstrate wave front curves dispersion behavior. The complicated features shown in Fig. 2(c) and to a lesser degree in Fig. 2(a) are due to multivalued behavior shown in Fig. 1 especially near $\xi = 4.2$ brought about by the presence of the higher modes.

In Fig. 3 variation of ratio of quasi-longitudinal wave speed c_{QL} to quasi-thermal wave speed c_{QT} with wave number ξ, for angles of incidence $\theta = 15^0$, $\theta = 30^0$, $\theta = 45^0$, $\theta = 60^0$, and $\theta = 90^0$, respectively.

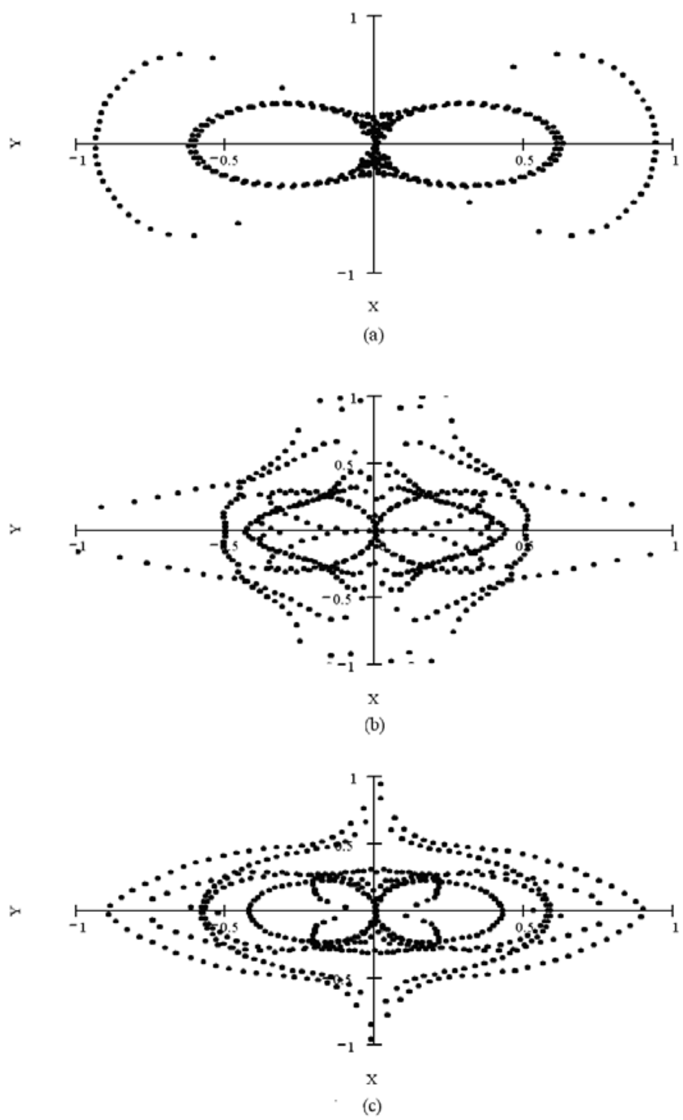

Figure 2: Wave-front curves for ξ and a AL/C/AL lay-up ($X = C\cos\theta, Y = C\sin\theta$).(a) $\xi = .000001$ mm^{-1}; (b) same as (a) repeated at $\xi = 1.5$ mm^{-1}; (c) same as (a) repeated at $\xi = 1.5$ mm^{-1}.

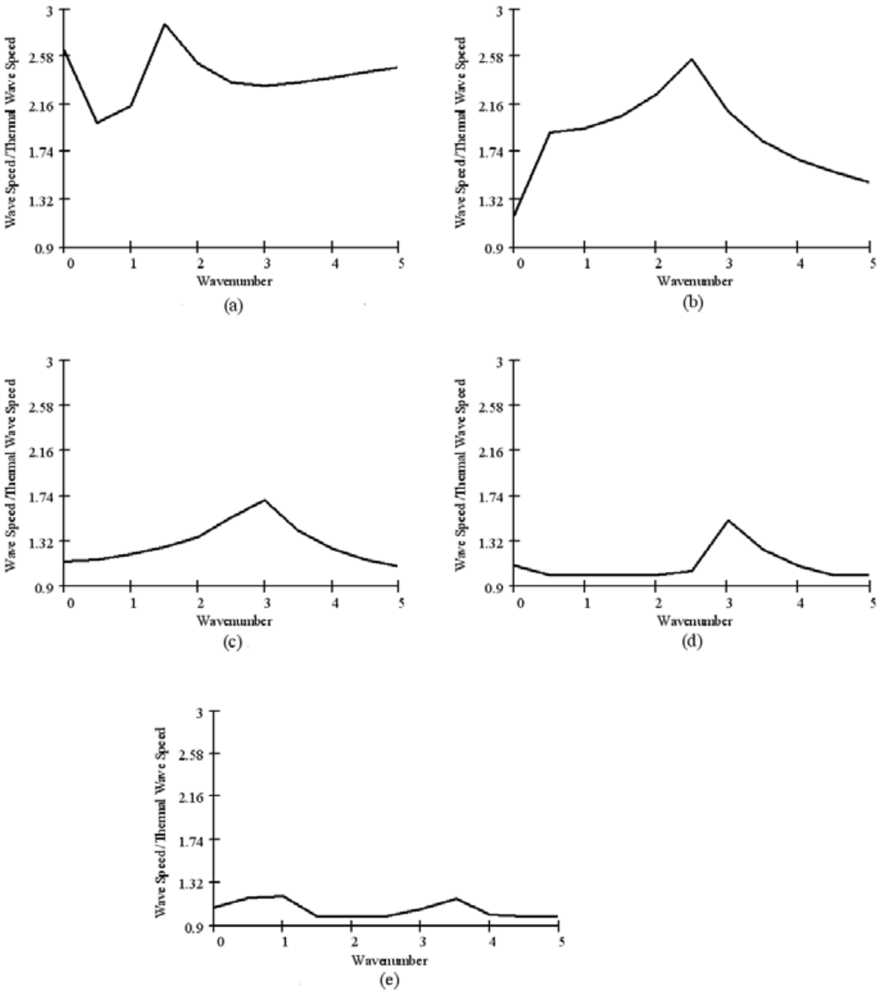

Figure 3: Variation of ratio of QL wave speed to T-Mode wave speed with wave-number ξ for various angles of incidence θ (a) 15^0, (b) 30^0, (c) 45^0, (d) 60^0 and (e) 90^0.

Fig 3(a) shows that quasi-longitudinal wave speed is 1.979 to 2.6999 times to that of quasi- thermal wave speed as

$$1.979 \leq \frac{c_{QL}}{c_{QT}} \leq 2.6999.$$

In Fig. 3(b), when $\theta = 30^0$, $1 \leq \frac{c_{QL}}{c_{QT}} \leq 2.692$ when $\xi \leq 9.3$, and

$$c_{QL} = c_{QT}, \text{ when } \xi \geq 9.3.$$

In Fig. 3(c), when $\theta = 45^0$, $1 \leq \frac{c_{QL}}{c_{QT}} \leq 1.697$, when $\xi \leq 6.3$, and

$$c_{QL} = c_{QT}, \text{ when } \xi \geq 6.3.$$

In Fig. 3(d), when $\theta = 60^0$, $1 \leq \frac{c_{QL}}{c_{QT}} \leq 1.518$, when $2.4 \leq \xi \leq 4.8$, and

$$c_{QL} = c_{QT}, \text{ when } \xi \leq 2.4 \text{ and } \xi \geq 4.8.$$

In Fig. 3(e) , when $\theta = 90^0$, $1 \leq \frac{c_{QL}}{c_{QT}} \leq 1.243$, when $\xi \leq 1.5$, and $2.7 \leq \xi \leq 3.9$

$$c_{QL} = c_{QT}, \text{ when } 1.5 \leq \xi \leq 2.7 \text{ and } \xi \geq 3.9.$$

It has been observed that wave velocity of quasi-longitudinal $c_{QL} \geq$ wave velocity of quasi-Thermal c_{QT} (T-Mode) for any angle of incidence and thermal relaxation time. Therefore quasi-longitudinal wave is found to be faster than thermal wave for layered media when the angle of incidence θ.

8 Conclusion

We have considered the problem of thermoelastic wave propagation in multilayered media in the context of generalized thermoelasticity. Analytical expressions have been derived that are easily adaptable to numerical illustrations. A thermoelastic plate consisting of an arbitrary number of layers, each layer possessing isotropic symmetry is chosen as a representative cell of medium. Characteristic equation for free harmonic thermoelastic waves in a multilayered plate and in periodic media are constructed from a repetition of the thermoelastic-layered plate. SH motion gets decoupled and is unaffected by thermal variation as in the case of coupled and classical theories of thermoelasticity. It has been observed that velocity of quasi-longitudinal (c_{QL}) wave $\geq$ velocity of quasi-Thermal (c_{QT}) wave, for any angle of incidence and also depends upon thermal relaxation time. It has also been found that quasi-thermal wave speed increases with angle of incidence and approaches to quasi-longitudinal (c_{QL}) wave speed. Further, the maximum value of quasi-longitudinal (c_{QL}) wave speed decreases with angle of incidence. This shows that quasi-longitudinal wave remain either faster or equal to thermal wave in layered media.

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Appendix 1

The coefficients of equation (3.5) are

$$P_1 = [3c_2 \sin^2 \theta - \{(1+\varepsilon_1)\omega^* \tau - 1\}c_2\zeta^2 - \zeta^2]/c_2$$

$$P_2 = [3c_2 \sin^4 \theta - \{1 + c_2 + c_2(1+\varepsilon_1)\omega^*\tau\}2\sin^2\theta\zeta^2$$

$$+\{1 + (c_2 + 1 + \varepsilon_1)\omega^*\tau\}\zeta^4]/c_2$$

$$P_3 = [\sin^4\theta + \omega^*\tau\zeta^4 - (1+\varepsilon_1)\omega^*\tau\sin^2\theta\zeta^2 - \sin^2\theta\zeta^2\}][c_2\sin^2\theta - \zeta^2]/c_2$$

Appendix 2

The pertinent data used in the computation are:

Aluminum epoxy:

$\lambda = 7.59 \times 10^{11}\text{dynes/cm}^2,\ \mu = 1.8910^{11}\text{dynes/cm}^2$
$C_e = 0.23\text{cal}/\ ^0C,\ \rho = 2.19\text{gm/cm}^3,\ K = 0.6 \times 10^{-2}\text{cal/cm}\sec\ ^0C$
$\varepsilon_1 = 0.073,\ \tau_0 = 6.131 \times 10^{-3}\text{s},\ \omega^* = 4.36 \times 10^{13}/\text{s}^{-1}.$

Carbon steel:

$\lambda = 9.3 \times 10^{11}\text{dynes/cm}^2,\ \mu = 8.4 \times 10^{11}\text{dynes/cm}^2,$
$C_e = 0.153\text{cal}/\ ^0C,\ \rho = 7.9\text{gm/cm}^3,\ K = 6.4 \times 10^{-2}\text{cal/cm}\sec\ ^0C,$
$\varepsilon_1 = 0.34,\ \omega^* = 6.24 \times 10^{12}/\text{s}^{-1},\ T_0 = 293.1\ ^0K$

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