HAIT Journal of Science and Engineering B, Volume x, Issue x, pp. xxx-xxx

Modal analysis of waveguide using method of moment

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Received 16 June 2007, accepted 15 November 2007

Abstract

Maxwell's equation for a waveguide whose medium has an inhomogeneous dielectric constant is formulated. The resulting modified Helmholtz equation is transformed to a matrix generalized eigen value problem using the method of moments. A complete numerical solution of Helmholtz equation for nonhomogeneous rectangular waveguide has been presented in this paper. This is implemented in MATLAB and the numerical values for the modes of propagation are obtained.

Keywords: Method of moment, anisotropic waveguide, propagation modes, eigen-vector, matrix equation.

1 Introduction

The method of moments (MOM) is arguably the most developed numerical technique [1] for solving electromegnatic (EM) scattering and radiation problems. Yet, the application of this method has been limited to resonant and lower frequencies.

A method of moments technique for the computational analysis of rectangular waveguide is presented elsewhere [1]. The waveguides are used to transmit electromagnetic signals. At high frequencies, this is the only practical way of transmitting electromagnetic radiation. In a waveguide, only special modes are transmitted and thus, the analysis of the eigenvalues and their eigenvectors becomes very important in the electromagnetic. The other advantage of the waveguides is that they can handle high power with low losses e.g., attenuation loss, transmission loss etc. The electromagnetic fields in these waveguides are given by the Maxwell equations. While these equations are linear, the analytical solutions of them can only be obtained in the spacial cases of boundary interface conditions. Due to this, electromagnetic applications have become more complex, however, the prediction of their performance, in general, would be important. Therefore, numerical methods have become a useful tool [2] to analyze the complex system of equations.

Wave equations govern two types of the wave propagation, namely, the propagation of mechanical waves and of the e1ectromagnetic waves. They can often be written either in the second order or in a system of the first order. Most of the literature focus on the numerical solution of the first order wave equations because the second order can be transformed into the first order [3]. But there are some cases that the second order wave equations need to be solved. There are several explicit methods with second order and higher order approximations [4, 5], but all the explicit methods have an upper band limit, for the time step size used. Present method can be used to find the solution of wave-equation for a waveguide having different region of permittivity and also for any shaped waveguide.

The available literature lacks with the numerical solution of the inhomogeneous Maxwell equations, even for a rectangular waveguides [6, 7]. Thus, the present work aims to fulfill this gap in the literature.

In this work, Maxwell equations for a rectangular waveguide, whose medium has an inhomogeneous dielectric constant, are formulated. This formulation results in the modified Helmholtz equations, which are transformed into a generalized eigen value problem using the method of moments. A complete numerical solution of this generalized eigen value problem for inhomogeneous rectangular waveguide has been obtained using MATLAB. The numerical values for modes of the propagation are obtained and compared with the practical values.

2 Problem statement

Assume that the waveguide is rectangular with dimensions a and b (Figure 1). The rectangular has been divided into four equal area regions. Let ϵ_1 and ϵ_2 be the permittivity for the lower $(0 \leq y \leq b/2)$ regions 1 $(0 \leq x \leq a/2)$ and 2 $(a/2 \leq x \leq a)$, respectively.

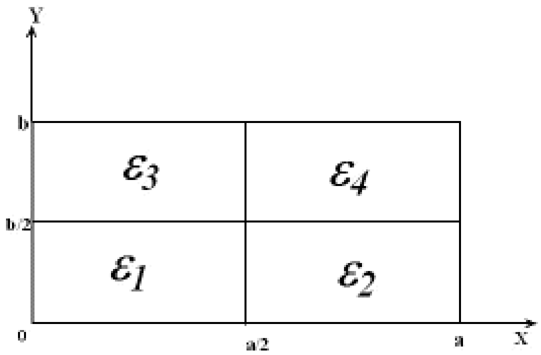

Figure 1: Rectangular waveguide cross section for four regions of permittivity

Similarly, ϵ_3 and ϵ_4 is the permittivity for the upper part $(b/2 \leq y \leq b)$ of the rectangle, i.e., regions 3 $(0 \leq x \leq a/2)$ and 4 $(a/2 \leq x \leq a)$, respectively. A cross-section for this type of waveguide is shown in Figure 1.

The wave equation is given by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2 \mu \epsilon + \gamma^2 \right) \psi(x, y) = 0 \tag{1}$$

subject to the boundary condition

$$\psi(0, y) = 0 = \psi(a, y) = \psi(x, 0) = \psi(x, b)$$

where $\varepsilon (= \varepsilon_0 . \varepsilon_r)$ and $\mu (= \mu_0 . \mu_r)$ are the permittivity and the permeability of the medium, respectively. The ε_r and μ_r are the relative permittivity and permeability of the medium. The permittivity and the permeability of the free space are given by

$$\varepsilon_0 = \quad 8.85 \times 10^{-12} F/m \tag{2a}$$

$$\mu_0 = \quad 4\pi \times 10^{-7} H/m \tag{2b}$$

and

$$\omega^2\mu\epsilon + \gamma^2 = h^2$$

Here h^2 is a constant, and the propagation constant γ is given as follow

$$\gamma = \alpha + j\beta$$

Where, α and β are the attenuation constant and the phase constant, respectively. The test function, $\psi(x, y)$, can be chosen as sin or cos function. By solving this equation, one can find the admissible values of propagation constant (γ).

3 The method of moments analysis

To apply the method of moments, the electromagnetic field is expressed via the potential orthonormalized modal function $\psi(x, y)$as follow

$$\psi(x, y) = \sum_{m,n=1}^{N} C_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \tag{3}$$

Substitution of equation (3) into equation (1) gives

$$\begin{aligned}
- & \sum_{m,n=1,2}^{N} [(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] C_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \\
+ & \omega^2 \mu \epsilon_(x, y) \sum_{m,n=1,2}^{N} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \\
+ & \gamma^2 \sum_{m,n=1,2}^{N} C_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) = 0
\end{aligned} \tag{4}$$

After multiplying equation (4) by

$$\sin(\frac{r\pi x}{a}) \sin(\frac{s\pi y}{b})$$

and integrating over the cross-section of the waveguide, we get

$$\begin{aligned}
& -(\frac{ab}{4})[(\frac{r\pi}{a})^2 + (\frac{s\pi}{b})^2] C_{rs} \\
& + \sum_{m,n=1,2}^{N} \omega^2 \mu C_{mn} \int_0^a \int_0^b \epsilon(x, y) \sin(\frac{m\pi x}{a}) \sin(\frac{r\pi x}{a}) \sin(\frac{s\pi y}{b}) \\
& \sin(\frac{n\pi y}{b}) dx.dy + \gamma^2 \frac{ab}{4} C_{rs} = 0
\end{aligned} \tag{5}$$

where $r, s = 1, 2 \cdots N$.

Let D be the diagonal matrix of size $N^2 \times N^2$ whose $[N(r-1)+s, N(r-1)+s]^{th}$ entry is

$$-\frac{ab}{4}[(\frac{r\pi}{a})^2) + (\frac{s\pi}{b})^2)]$$

and let

$$A[r,s/m,n] = \int_0^a \int_0^b \epsilon_{x,y} \sin(\frac{n\pi y}{b}) \sin(\frac{r\pi x}{a}) \sin(\frac{s\pi y}{b}) \sin(\frac{m\pi x}{a}) dx.dy$$

Now, we break this integration into four parts (of the rectangle, see Fig. 1)

$$\begin{aligned} A[r,s/m,n] &= \int_0^{a/2} \int_0^{b/2} \epsilon_1 \sin(\frac{n\pi y}{b}) \sin(\frac{r\pi x}{a}) \sin(\frac{s\pi y}{b}) \sin(\frac{m\pi x}{a}) dx.dy \\ &+ \int_{a/2}^{a} \int_0^{b/2} \epsilon_2 sin(\frac{n\pi y}{b}) sin(\frac{r\pi x}{a}) sin(\frac{s\pi y}{b}) sin(\frac{m\pi x}{a}) dx.dy \\ &+ \int_0^{a/2} \int_{b/2}^{b} \epsilon_3 \sin(\frac{n\pi y}{b}) \sin(\frac{r\pi x}{a}) \sin(\frac{s\pi y}{b}) \sin(\frac{m\pi x}{a}) dx.dy \\ &+ \int_{a/2}^{a} \int_{b/2}^{b} \epsilon_4 \sin(\frac{n\pi y}{b}) \sin(\frac{r\pi x}{a}) \sin(\frac{s\pi y}{b}) \sin(\frac{m\pi x}{a}) dx.dy \end{aligned}$$

Let

$$A[r,s/m,n] = I1[r,s/m,n] + I2[r,s/m,n] + I3[r,s/m,n] + I4[r,s/m,n]$$

where

$$\begin{aligned} I1[r,s/m,n] = & \frac{ab\epsilon_1}{4\pi^2} \\ & \times [\frac{\sin(r-m)\pi/2 \sin(s-n)\pi/2}{(r-m)(s-n)} - \frac{\sin(r-m)\pi/2 \sin(s+n)\pi/2}{(r-m)(s+n)} \\ & - \frac{\sin(r+m)\pi/2 \sin(s-n)\pi/2}{(r+m)(s-n)} + \frac{\sin(r+m)\pi/2 \sin(s+n)\pi/2}{(r+m)(s+n)}] \\ & \text{for } r \neq m, s \neq n \end{aligned} \tag{6}$$

and if r=m and s=n then

$$I1[r,s/m,n] = \epsilon_1 \frac{ab}{16} \tag{7}$$

and

$$\begin{aligned}
I2[r,s/m,n] = & \frac{ab\epsilon_2}{4\pi^2} \\
& \times[\frac{\sin(s-n)\pi/2\sin(r+m)\pi/2}{(s-n)(r+m)} - \frac{\sin(s-n)\pi/2\sin(r-m)\pi/2}{(s-n)(r-m)} \\
& -\frac{\sin(r+m)\pi/2\sin(s+n)\pi/2}{(r+m)(s+n)} + \frac{\sin(s+n)\pi/2\sin(r-m)\pi/2}{(s+n)(r-m)}] \\
& \text{for } r \neq m, s \neq n
\end{aligned} \tag{8}$$

and if r=m and s=n then

$$I2[r,s/m,n] = -\epsilon_2 \frac{ab}{16} \tag{9}$$

Similarly,

$$\begin{aligned}
I3[r,s/m,n] = & \frac{ab\epsilon_3}{4\pi^2} \\
& \times[\frac{\sin(r-m)\pi/2\sin(s+n)\pi/2}{(r-m)(s+n)} - \frac{\sin(r-m)\pi/2\sin(s-n)\pi/2}{(r-m)(s-n)} \\
& -\frac{\sin(r+m)\pi/2\sin(s+n)\pi/2}{(r+m)(s+n)} + \frac{\sin(r+m)\pi/2\sin(s-n)\pi/2}{(r+m)(s-n)}] \\
& \text{for } r \neq m, s \neq n
\end{aligned} \tag{10}$$

and if r=m and s=n then

$$I3[r,s/m,n] = -\epsilon_3 \frac{ab}{16} \tag{11}$$

and

$$\begin{aligned}
I4[r,s/m,n] = & \frac{ab\epsilon_4}{4\pi^2} \\
& \times[\frac{\sin(s+n)\pi/2\sin(r+m)\pi/2}{(s+n)(r+m)} - \frac{\sin(s-n)\pi/2\sin(r+m)\pi/2}{(s-n)(r+m)} \\
& -\frac{\sin(r-m)\pi/2\sin(s+n)\pi/2}{(r-m)(s+n)} + \frac{\sin(s-n)\pi/2\sin(r-m)\pi/2}{(s-n)(r-m)}] \\
& \text{for } r \neq m, s \neq n
\end{aligned} \tag{12}$$

and if r=m and s=n then

$$I4[r,s/m,n] = \epsilon_4 \frac{ab}{16} \tag{13}$$

Now

$$I[r,s/m,n] = I1[r,s/m,n] + I2[r,s/m,n] + I3[r,s/m,n] + I4[r,s/m,n] \quad (14)$$

Substituing the values of I_1, I_2, I_3 and I_4 into equation (14). If r≠ m and s≠n then we get

$$\begin{aligned} I[r,s/m,n] &= \frac{ab(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)}{4\pi^2} \\ &\left[\frac{\sin(s-n)\pi/2}{s-n} - \frac{\sin(s+n)\pi/2}{s+n}\right] \\ &\left[\frac{\sin(r-m)\pi/2}{r-m} - \frac{\sin(r+m)\pi/2}{r+m}\right] \end{aligned} \quad (15)$$

and if r=m and s=n then

$$I[r,s/m,n] = \frac{ab}{16}[\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4] \quad (16)$$

Here we assume that ϵ_1=4, ϵ_2=2, ϵ_3=1, ϵ_4=3 and

$$\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 = 4 \quad (17)$$

Now by substituting the value of equation (17) into equations (15) and (16), we have

$$\begin{aligned} I[r,s/m,n] &= \frac{ab}{\pi^2}\left[\frac{\sin(s-n)\pi/2}{s-n} - \frac{\sin(s+n)\pi/2}{s+n}\right] \\ &\left[\frac{\sin(r-m)\pi/2}{r-m} - \frac{\sin(r+m)\pi/2}{r+m}\right] \\ \text{for } r &\neq m, s \neq n \end{aligned} \quad (18)$$

and at r=s and m=n we have

$$I[r,s/m,n] = \frac{ab}{4} \quad (19)$$

Now, let A denote an $N^2 \times N^2$ matrix whose $[N(r-1)+s, N(m-1)+n]^{th}$ entry equals $A[r,s/m,n]$, then the system of linear equation will be expressed as

$$\omega^2 \mu Ac + Dc + \gamma^2(\frac{ab}{4})c = 0 \quad (20)$$

Here

$$\omega^2 \mu A + D = B \quad (21)$$

Now equation (20) can be written as

$$Bc + \lambda c = 0 \tag{22}$$

now the admissible values of γ are given by

$$\gamma_k = \pm 2\sqrt{\frac{-\lambda_k}{ab}} \tag{23}$$

$k = 1, 2, 3 \cdots N^2$. where $\lambda_k : k = 1, 2, 3 \cdots N^2$ are the eigen values of the matrix

$$B = \omega^2 \mu A + D \tag{24}$$

4 Simulation results and discussion

The simulations have been carried out for the frequency, $\omega = 2\pi \times 20 \times 10^9$ rad/sec and for $N = 8, 16, 24$. The values of γ has been calculated from the eigen values of the matrix, $B = \omega^2 \mu A + D$. The approximate theoretical values of the modes are also calculated by averaging the values of the permittivities throughout the waveguide cross-section. The attenuation constant is very low in the calculated values, which is always desirable. Table 1 shows a comparison between the theoretical and calculated values. The two values (theoretical and calculated) are seen to be very close to each other in Table 1. This close agreement in between these values clearly show a validity of the present solution procedure.

In this work, we have considered the structure with rectangular cross-section. However, this method is not limited to any structure and can be applied to any kind of structure, as shown in the Fig. 2. Similar procedure can be followed to find the solution of the other (non-rectangular) structures.

In summary, we have started by writing down the modified Helmholtz equation for the propagation of TM modes in a rectangular waveguide with piecewise constant permittivity. The solution $\psi(x, y)$ has been expanded in terms of sine waves. Then the method of moments is applied to reduce the original infinite dimensional problem to a finite dimensional matrix generalized eigenvalue problem. The size of the matrices depend on the number of sine-waves (N) used in the the expansion. As this number of sine-waves grows, the size of the matrices also grow. In this paper, the propagation modes are obtained using MATLAB for three sample values of N (see Table 1). Numerical simulations show that the modes converge quite rapidly when $N > 15$. It would be quite difficult to obtain a theoretical solution of the modes as a function of N. This would involve determining the behavior

Table 1: Comparison between the theoretical and calculated values of γ.

γ(theoretical values)	γ(calculated values)		
	N=8	N=16	N=24
931.3i	937.26i	957.43i	964.83i
933.5i	937.26i	957.43i	964.83i
943.6i	937.25i	957.42i	964.83i
956.9i	937.24i	957.41i	964.82i
956.9i	937.23i	957.41i	964.82i
958.5i	937.22i	957.42i	964.81i
962.6i	937.20i	957.39i	964.80i
962.6i	937.18i	957.38i	964.79i
1001.6i	937.26i	957.40i	964.79i
1006.4i	937.25i	957.41i	964.78i
1185.1i	937.25i	957.37i	964.78i
1215.4i	937.24i	957.36i	964.77i
1401.1i	937.23i	957.35i	964.76i
1401.1i	937.21i	957.35i	964.75i
1823.1i	937.19i	957.37i	964.74i
1823.1i	937.18i	957.35i	964.74i

of the eigenvalues of the matrix as a function of its size. Specifically one would need to address a problem of the following kind.

Let $\pounds_1$ and $\pounds_2$ be two linear partial differential operators and ϕ_n $n = 1, 2, 3 \cdots n$ is a sequence of test functions. Then determine the behavior of the solution, of the equation

$$f_N(\lambda) = det(A_N - \lambda B_N) = 0$$

as N increases where

$$A_N = \left(\left(\int_D \phi_n(x, y)\pounds_1\phi_m(x, y)dxdy\right)\right)$$

and

$$B_N = \left(\left(\int \phi_n(x, y)\pounds_2\phi_m(x, y)dxdy\right)\right)$$

As far as known to us, none of the existing methods can be used to solve such type of problem.

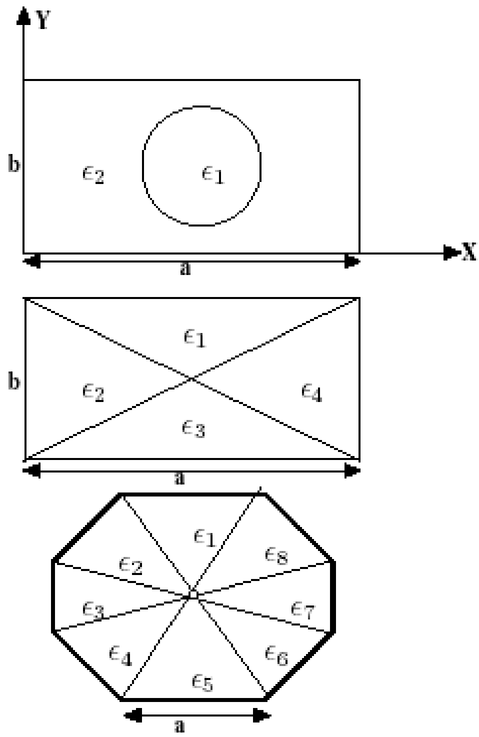

Figure 2: Different possible shapes of waveguides in which present method can be applied to obtain the propogation modes.

5 Conclusion

In this work, electromagnetic wave propagation analysis has been carried out numerically. The Maxwell equations have been formulated for a inhomogeneous rectangular waveguide. The resulting modified Helmholtz equations are transformed into a generalized eigen values problem using the method of moments. The eigen values of the system of linear equations are obtained by using MATLAB. These eigen values are used to obtain the wave propagation coefficient (γ). A good correspondence was seen between the theoretical and practical values. Attenuation is very low and the propagation is high. We could calculate the modes by using any numerical technique like finite element method, finite difference method, but the reason for using method of moment here is due to it's simplicity. Microwave elements constructed using waveguide sections are pretty common. When allowance is made for variable dielectric constant then one attains a greater degree of freedom leading to a wide set of circuit parameters. In other words, a wider range of impedances can be generates using the same waveguide section by appropriately adjusting the distribution of the dielectric inside it.

The authors gratefully acknowledge Prof. Raj Senani for his constant encouragement and provision of facilities for this research work.

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