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Satellite tracking using a second-order stochastic nonlinear filter

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Abstract

In the theory of classical mechanics, the two-body central forcing problem is formulated as a system of the coupled nonlinear second-order deterministic differential equations. The uncertainty introduced by the small, unmodelled stochastic acceleration, is not assumed in the particle dynamics. The small, unmodelled stochastic acceleration produces an additional random force on a particle. Estimation algorithms of a two-body dynamics, without introducing the stochastic perturbation, may cause inaccurate estimation of a particle trajectory. In particular, this paper examines the effect of the stochastic acceleration on the motion of the orbiting satellite, and subsequently, the stochastic estimation algorithm is developed by deriving the evolutions of conditional means, and conditional variances for estimating the state of the satellite. By linearizing the stochastic differential equations about the mean of the state vector using first-order approximation, the mean trajectory of the resulting first-order approximated stochastic differential model does not preserve the perturbation effect felt by the orbiting satellite; only the variance trajectory includes the perturbation effect. For this reason, the effectiveness of the perturbed model is examined on the basis of the second-order approximations of the system nonlinearity. The theory of the nonlinear filter of this paper is developed using the Kolmogorov forward equation 'between the observation' and a functional difference equation for the conditional probability density 'at the observation'. The effectiveness of the second-order nonlinear filter is examined on the basis of its ability to preserve perturbation effect felt by the orbiting satellite and the *signal-noise ratio.* The Kolmogorov forward equation, however, is not appropriate for numerical simulations, since it is the equation for the evolution of the conditional probability density. Instead of the Kolmogorov equation, one derives the evolutions for the moments of the state vector, which in our case consists of positions and velocities of the orbiting satellite. Even these equations are not appropriate for the numerical simulations, since they are not closed in the sense that computing the evolution of a given moment involves the knowledge of higher-order moments. Hence we consider the approximations to these moment evolution equations. Simulation results are introduced to demonstrate the usefulness of an analytic theory developed in this paper.

Keywords: Satellite tracking, stochastic differential equation, Brownian-motion process, Fokker-Planck Kolmogorov equation, mean, variance.

1 Introduction

Since 1960, there have been important works on filtering. Kalman filtering is one of the filtering approaches, having applicability in various fields of signal processing and control. This filter is based upon a given linear model for the states of the system and another linear model for the observation process. The linearity condition on the state-cum-observation model severely restricts the applications of the Kalman filter. For the filtering of non-linear systems, extended Kalman filtering was greatly investigated, by the use of the first-order approximation of the non-linear stochastic model, about the conditional means of the states [1-4]. However, the extended Kalman filter suffers from bad initializations. The extended Kalman filter is not accurate enough because of the inaccurate representation of the non-linearities by the linearized model. Linearizing approximations made during the evolutions of the conditional means, and variances for the extended Kalman filter do not allow non-linear perturbation effects, which account for the modelling error. Thus, the system theorists started looking for approximate filters that permit the closed-form implementation and preserve some of the important qualitative characteristics of the exact filter. These approximations are based upon the Taylor expansions of the non-linear functions of the stochastic differential model, about the conditional mean of the state vector. The second-order filter is one of the approximations. Locally optimal filters were considered by Pugachev [3-5] who derived the conditionally optimal nonlinear filter by parametric optimization of a nonlinear filter.

To realize Pugachev's method, one needs to know the joint characteristic function of the system state and the filter estimate. This function can be obtained by solving the special non-random functional equation, which is rather complicated, and takes much effort to obtain the desired solution, especially for the multidimensional case. The derivative-free filters (e.g. UKF, PF) have been proposed in [6, 7]. UKF performs a stochastic linearization through the use of a weighted statistical linear regression process. The performance of UKF depends on the higher order moments of the distribution which is scenario-dependent. In addition, when the initial condition of the filter is poor, UKF may take a long time to settle [8]. One of the difficulties encountered is regarding the choice of the approximate filter for a given problem [9-13]. The general philosophy in choosing the adequate filter is dictated by two factors: one, the complexity of the filter in the sense of the number of moments required to be computed, and two, the variance, i.e. we choose an order of the approximation so that variances of the state vector are not too large as that would imply that the conditional means of the state vector are not too random.

In what follows, we briefly survey some of the important developments in non-linear filtering. Mehra examined the accuracy of the extended Kalman filter in different co-ordinate systems for the real-time estimation or tracking of ballistic Re-entry Vehicle (RV) from its radar observations [14]. Itô has developed and analyzed Gaussian filter for the non-linear stochastic systems, in which conditional probability density is approximated by a sum of Gaussian distributions [15]. Shaikin established a finite- dimensional approximation for the optimum filtering equations of the Markov diffusion process described by a stochastic system, where an approximate filter is derived as Kushner-Stratonovich equation with finite Peano series [16]. Kushner simulated a Gaussian-type approximate filter for a second-order system (Vander Pol equation) with linear measurements [11]. Third-order moments were neglected, and fourth-order moments were approximated, using Gaussian assumptions. Shrish et al. derived the finite-dimensional filter for the restrictive stochastic model [17]. An accurate modelling procedure coupled with effective filtering algorithm can increase the efficacy of the estimation procedure [18]. Accurate modelling procedure is accomplished by introducing the small, unmodelled stochastic acceleration felt by the orbiting satellite. The problem of accurate modelling procedure has received attention in literature. In [19, 20], the authors have examined the effect of the stochastic acceleration on the dynamical model of a spacecraft trajectory. In Scheeres' analysis, the integration of the Ricatti equation was carried out for orbit uncertainty of a one-dimensional force-free motion incorporating a stochastic analysis with correlation time t. The evolutions of variances were derived using the Ricatti equation. However, the problem of estimating the states of the stochastically perturbed orbiting particle from noisy measurements, especially accounting for the 'stochastic acceleration' in equation of the motion, has not been examined in detail. In this paper, the stochastic differential equation formalism for the non-linear problem of concern here, would be the subject of investigation.

The purpose of this paper is to develop a non-linear continuous-discrete time filter, for the two-body problem (satellite-earth system), formulated in the form of a stochastic non-linear differential system. The performance of the non-linear filter is examined, on the basis of its ability to preserve perturbation effect felt by the orbiting satellite and the signal-noise ratio. This paper utilizes the Kolmogorov forward equation for the mean and variance evolutions 'between the observations' and the functional difference equation for the conditional probability density 'at the observation'. This paper discusses the stochastically perturbed two-body problem, treating the effect of the stochastic acceleration as the 'state-parameter-independent noise perturbation'.

This paper is organized as follows: Section 2 deals with a stochastically perturbed non-linear model with Brownian motion inputs and nonlinear measurement model for the satellite tracking problem. In Section 3, the evolutions of conditional means, and conditional co-variances are derived for the stochastic non-linear differential model of the satellite tracking problem. In Section 4, simulation results are given to show the feasibility of the analysis results derived in Section 3. Concluding remarks are presented in Section 5.

2 Equations of motion

Reference [14] offers a good discussion of how to handle re-entry vehicle (or satellite) target models (a topic that, unfortunately, is usually absent from other discussions concerned with the same type of target tracking applications) and provides a derivation of the particulars from first principles as well as providing an accounting and motivation for use of the various coordinate systems. A rigorous analysis of other important modelling considerations are treated in [21]. we follow a similar path in the choice of mathematical models used here.

In our investigation, a Keplerian trajectory is introduced within a detailed simulation of the exoatmospheric target motion to include the effect of an inverse square pull of gravity and use is made of a more sophisticated filter model to handle tracking in the presence of these inverse square nonlinearities and state independent noises modelled using Brownian motion processes. Use of this more exacting methodology to represent gravity more realistically requires that we depart from just the use of simplified covariance analysis (essentially corresponding to evaluation of a Cramer-Rao lower bound for the estimation objective in the exoatmospheric regime of no process noise being present as used in earlier investigations [22, 23]), and instead now requires that we incorporate full nonlinear filtering techniques (and the associated standard approximations).

The orbital elements of a satellite in three dimensional space is determined by its position and velocity vectors at a given epoch, and the two body forces acting on it. Once the position and velocity vectors are obtained in inertial frame, the orbital motion of the satellite is described by the following equation

$$\frac{d}{dt}\begin{pmatrix} x_t \\ y_t \\ z_t \\ v_{x_t} \\ v_{y_t} \\ v_{z_t} \end{pmatrix} = \begin{pmatrix} v_{x_t} \\ v_{y_t} \\ v_{z_t} \\ \frac{-GMx_t}{(x_t^2+y_t^2+z_t^2)^{3/2}} \\ \frac{-GMy_t}{(x_t^2+y_t^2+z_t^2)^{3/2}} \\ \frac{-GMz_t}{(x_t^2+y_t^2+z_t^2)^{3/2}} \end{pmatrix} \tag{1}$$

where $(x_t, y_t, z_t)^T$ represents the position vector, $(v_{x_t}, v_{y_t}, v_{z_t})^T$ represents the velocity vector in Earth-centered inertial (ECI) frame (the ECI frame as the one in which x-axis is along the first point of aries, z-axis along the earth's spin axis and y-axis completes the orthogonal triad of the right handed system), G is the gravitational constant and M is the mass of the central reference body. Accurate modelling procedure is accomplished by introducing small perturbation effect felt by the orbiting satellite, which leads to the stochastic framework. Thus, our model is different and better than that of the deterministic dynamics. The approach of this paper is a direct demonstration of the application of the motion of a satellite in an orbit. An example of how random disturbances arise in the motion of an orbiting satellite is via the presence of distributed matter in the vicinity of the field of motion like meteors, comets and asteroids. These distributed matter may also be in motion, and the gravitational force exerted by this distributed matter on the satellite. Mann et. al. have described the comets and asteroids as the major sources of the dust production [25]. Dust collisional fragmentation, sublimation, radiation pressure acceleration and rotational busting are the major causes of the dust loss processes. In their paper, they talk about the recent observations of the Sun-grazing comets. The dust particles are small and randomly distributed. The randomly distributed dust produces additional random force on the orbiting satellite. It is always strived that the estimated trajectory be close to the actual trajectory. The accurate modelling procedure increases the effectiveness of the estimated trajectory. The accurate modelling procedure is accomplished by introducing small perturbations felt by the orbiting satellite. For these reasons, it is important to introduce the dust perturbations on the orbiting satellite. It is a well known fact that random disturbances in the motion of a particle can be modelled as Gaussian white noise to a good degree of accuracy. Gaussianity of the random disturbance is a consequence of the central limit theorem according to which, if the random force is the cumulative sum of a very large number of small, independent random effects, then it has approximately a Gaussian law. The white property is a consequence of the fact that the disturbances at different times are independent, for example, the molecular kicks on a polar particle in a warm liquid at different times are independent owing to rapid random independent motions of the molecules. After introducing the effect of stochastic acceleration into the motion of the orbiting satellite, the equation of motion in the stochastic differential equation formalism is stated as

$$d\xi_t = f(\xi_t)dt + G(t)dB_t \tag{2}$$

where $\xi_t = \begin{pmatrix} x_t \\ y_t \\ z_t \\ v_{x_t} \\ v_{y_t} \\ v_{z_t} \end{pmatrix}$, $f(\xi_t) = \begin{pmatrix} v_{x_t} \\ v_{y_t} \\ v_{z_t} \\ \frac{-GMx_t}{(x_t^2+y_t^2+z_t^2)^{3/2}} \\ \frac{-GMy_t}{(x_t^2+y_t^2+z_t^2)^{3/2}} \\ \frac{-GMz_t}{(x_t^2+y_t^2+z_t^2)^{3/2}} \end{pmatrix}$,

$G(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_{v_x} & 0 & 0 \\ 0 & \sigma_{v_y} & 0 \\ 0 & 0 & \sigma_{v_z} \end{pmatrix}$, and $dB_t = \begin{pmatrix} dB1_t \\ dB2_t \\ dB3_t \end{pmatrix}$

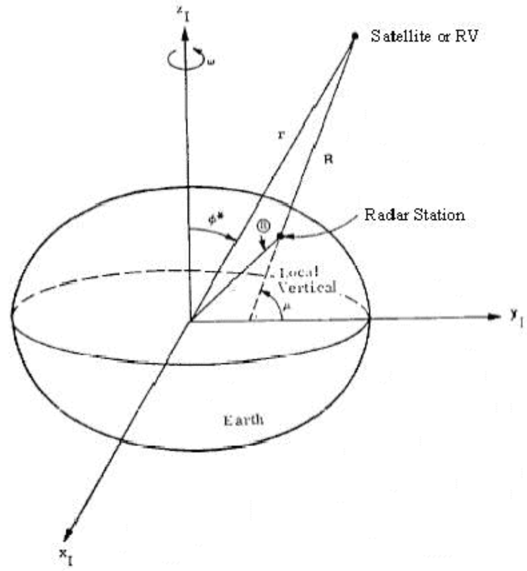

Figure 1: Radar-satellite geometry (see [8]).

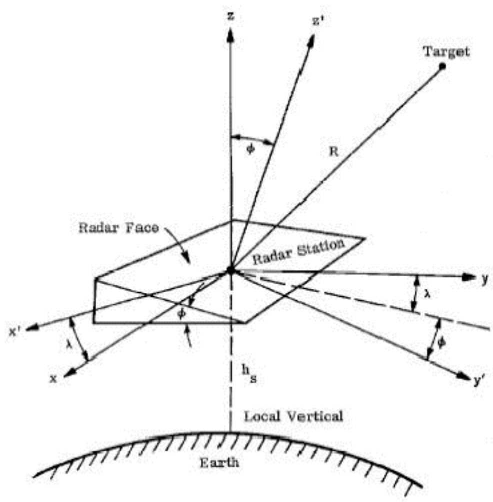

Figure 2: Coordinate systems centered at the radar (see [8]).

In order to use the ECI coordinate frame and system equations, the measurement equations are of a form addressed below. The measurement equations used for the present sensor model are as obtained from Figs. 1 and 2. The resulting sensor measurements in terms of range R, and the direction cosines u and v, to the target RV are

$$R = \sqrt{x'^2 + y'^2 + z'^2} \tag{3}$$

$$u = \frac{x'}{R} \tag{4}$$

$$v = \frac{y'}{R} \tag{5}$$

where x', y', and z' are as in Fig. 2 (and are to be defined next). In Fig. 2, the local coordinates x, y, z are located at the center of the sensor face in the plane of the array. In this coordinate system, z is directed along the local vertical and x and y lie in the horizontal plane, with x pointing East and y pointing North. From ([14], sec. II), these local level coordinates x, y, z can be represented in terms of x', y', z' coordinates via the following transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where

$$T = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \cos\phi\sin\lambda & \cos\phi\cos\lambda & -\sin\phi \\ \sin\phi\sin\lambda & \sin\phi\sin\lambda & \cos\phi \\ \end{pmatrix}$$

as the appropriate change of coordinates corresponding to the rotation depicted in Fig. 2, where the above parameters λ and ϕ are also defined in Fig. 2. The coordinates x', y', z' are oriented so that z' is normal to the face of the sensor array, and y' lies on the face of the array, and x' lies along the intersection of the sensor face and the horizontal plane.

The above received sensor signal-processed measurement can be represented in terms of the measurement of target range (as appropriate for a radar or other active sensor if not range-denied due to jamming), elevation, and azimuth as, respectively [24]

$$r_{t_k} = \sqrt{x_{t_k}^2 + y_{t_k}^2 + z_{t_k}^2} \tag{6}$$

$$E_{t_k} = \arctan\left[\frac{z_{t_k}}{\sqrt{x_{t_k}^2 + y_{t_k}^2}}\right] \tag{7}$$

$$A_{t_k} = \arctan\left[\frac{x_{t_k}}{y_{t_k}}\right] \tag{8}$$

where the length in (6) is identical to the length in (3) since the transformation T is a rotation (and as such is an orthogonal transformation which preserves lengths). The expressions of (6)-(8) correspond to the following measurement equation:

$$\zeta(t_k) = h(\xi_{t_k}) + \nu(t_k) \tag{9}$$

or

$$\zeta(t_k) = \begin{pmatrix} r_{t_k} \\ E_{t_k} \\ A_{t_k} \end{pmatrix} + \nu(t_k) \tag{10}$$

where $\nu(t_k)$ is an m-vector ($m = 3$) Gaussian white measurement noise, $\nu(t_k) \sim N(0, R_k), R_k > 0$. ξ_{t_0}, $\{B_t\}$ and $\{\nu_k\}$ are assumed independent. The Gaussian white measurement noise, $\nu(t_k)$ has a covariance that is of the form [24].

$$R_k = \begin{pmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_E^2 & 0 \\ 0 & 0 & \frac{\sigma_E^2}{cos^2(E)} \\ \end{pmatrix} \tag{11}$$

3 A continuous-discrete filter

In case of less observation rates and lack of communication of the orbiting satellite with measurement station, the problem of estimating the states of the orbiting satellite i.e. accurate positioning of the satellite reduces to the prediction algorithm *between the observations*. Here, first we explain the analysis for scalar case and subsequently, extended to vector case. The whole analysis can be divided into two parts: first 'between the observations' and secondly, 'at the observations'.

3.1 Between the observations

Mean and Variance evolutions are

$$\frac{d\hat{\xi}_t}{dt} = \hat{f}(\xi_t, t) \tag{12}$$

$$\frac{d\hat{P}_t}{dt} = 2(\widehat{\xi_t f} - \hat{\xi}_t \hat{f}) + \widehat{g^2 Q} \tag{13}$$

In matrix vector format, the above equations become

$$\frac{d\hat{\xi}_t}{dt} = \hat{f}(\xi_t, t) \tag{14}$$

$$\frac{d\hat{P}_t}{dt} = (\widehat{\xi_t f^T} - \hat{\xi}_t \hat{f}^T) + (\widehat{f \xi_t^T} - \hat{f}\hat{\xi}_t^{\,T}) + \widehat{GQG^T} \tag{15}$$

where the state vector ξ_t is $n \times 1$, system nonlinearity $f(\hat{\xi}_t, t)$ is $n \times 1$ and $G(t)$ is $n \times r$ for r-dimensional Brownian motion process. For the numerical simulations, component version is more convenient form. For this reason, we utilize the component versions of the vector case for the analysis. In component version, the above become

$$d\hat{\xi}_{i_t} = \hat{f}_i(\xi_t, t)dt \tag{16}$$

$$(dP_t)_{ij} = \big((\widehat{\xi_i f_j} - \hat{\xi}_i \hat{f}_j) + (\widehat{f_i \xi_j} - \hat{f}_i \hat{\xi}_j) + (\widehat{GQG^T})_{ij}\big)dt \quad t_{k-1} \le t < t_k \tag{17}$$

where

$$\hat{\xi}_{i_t} = \mathbb{E}(\xi_{i_t} \mid \zeta_{t_{k-1}}) and P_{ij} = \mathbb{E}\Big((\xi_{i_t} - \mathbb{E}(\xi_{i_t} \mid \zeta_{t_{k-1}}))(\xi_{j_t} - \mathbb{E}(\xi_{j_t} \mid \zeta_{t_{k-1}})) \mid \zeta_{t_{k-1}}\Big).$$

The above equations describe the evolutions of the exact filter, which is infinite dimensional. Because of infinite dimensionalities involved, the numerical implementation is not possible. The mean trajectory for the dust-perturbed satellite using first-order approximation does not include variance term in mean evolution. The term $\widehat{GQG^T}$ in variance evolution accounts for the stochastic perturbation felt by the orbiting satellite ([1] p.363). For this reason, the first order approximation does not preserve perturbation effects in mean evolution. On the other hand, the variance evolution using first-order approximation for the dust-perturbed model includes perturbation effects i.e. $\widehat{GQG^T}$. The main interest in first-order approximate model is owing to computational simplicity as well as it representing a 'nearly' linear yet nonlinear system [26].

In order to account for the stochastic perturbation in the mean evolution, we use the second-order approximation in the mean evolution. The second-order approximation includes the second-order partials of the system nonlinearity $f(\xi_t, t)$, and variance terms in the mean trajectory, which leads to better estimation of the trajectory [27]. For this reason, we introduce the second-order Taylor expansions of the system nonlinearity $f(\xi_t, t)$, measurement nonlinearity $h(\xi_t, t)$ about the conditional mean. After second-order approximation, we get

$$d\hat{\xi}_{t_i} = f_i(\hat{\xi}_t, t)dt + \frac{1}{2}\sum_{j,k=1}^{n} P_{jk}\frac{\partial^2 f_i(\hat{\xi}, t)}{\partial \xi_j \partial \xi_k}dt \tag{18}$$

$$(dP_t)_{ij} = \Big(\sum_{k=1}^{n} P_{ik}\frac{\partial f_j(\hat{\xi}, t)}{\partial \xi_k} + \sum_{k=1}^{n} P_{jk}\frac{\partial f_i(\hat{\xi}, t)}{\partial \xi_k} + (\widehat{GQG^T})_{ij}\Big)dt \tag{19}$$

The discussion on equation (18) and (19) using the 'Kolmogorov forward equation' is given separately in 'appendix A' of the paper. This paper discusses 6-dimensional diffusion equation. As a result of this, the size of the conditional mean vector of the process ξ_t is 6×1 and the number of entries in conditional variance matrix is $6 + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 21$. Equation (18) and (19) in conjunction with equation (2) for the nonlinear satellite tracking problem considered in this paper, we have the following mean and variance evolutions:

$$d\hat{x}_t = \hat{v}_{x_t}dt \tag{20}$$

$$d\hat{y}_t = \hat{v}_{y_t}dt \tag{21}$$

$$d\hat{z}_t = \hat{v}_{z_t}dt \tag{22}$$

$$\begin{aligned} d\hat{v}_{x_t} \quad = \quad & \frac{-GM\hat{x}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{3/2}}dt + \\ & \Bigg(P_{x_tx_t}\frac{GM\hat{x}_t(9\widehat{y^2}_t + 9\widehat{z^2}_t - 7\widehat{x^2}_t)}{2(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{7/2}} - P_{x_ty_t}\frac{3GM\hat{y}_t(4\widehat{x^2}_t - \widehat{y^2}_t - \widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{7/2}} \\ & -P_{x_tz_t}\frac{3GM\hat{z}_t(4\widehat{x^2}_t - \widehat{y^2}_t - \widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{7/2}} + P_{y_ty_t}\frac{GM\hat{x}_t(\widehat{x^2}_t - 4\widehat{y^2}_t + \widehat{z^2}_t)}{2(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{7/2}} \\ & -P_{y_tz_t}\frac{15GM\hat{x}_t\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{7/2}} + P_{z_tz_t}\frac{GM\hat{x}_t(\widehat{x^2}_t + \widehat{y^2}_t - 4\widehat{z^2}_t)}{2(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{7/2}}\Bigg)dt \end{aligned} \tag{23}$$

$$
\begin{aligned}
d\hat{v}_{y_t} = {} & \frac{-GM\hat{y}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{3/2}}dt + \\
& \Bigg(P_{x_tx_t}\frac{GM\hat{y}_t(\widehat{y^2}_t+\widehat{z^2}_t-4\widehat{x^2}_t)}{2(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} - P_{x_ty_t}\frac{3GM\hat{x}_t(4\widehat{y^2}_t-\widehat{x^2}_t-\widehat{z^2}_t)}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} \\
& -P_{x_tz_t}\frac{15GM\hat{x}_t\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} + P_{y_ty_t}\frac{GM\hat{y}_t(9\widehat{x^2}_t+9\widehat{z^2}_t-7\widehat{y^2}_t)}{2(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} \\
& -P_{y_tz_t}\frac{3GM\hat{z}_t(4\widehat{y^2}_t-\widehat{x^2}_t-\widehat{z^2}_t)}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} + P_{z_tz_t}\frac{GM\hat{y}_t(\widehat{x^2}_t+\widehat{y^2}_t-4\widehat{z^2}_t)}{2(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}}\Bigg)dt
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
d\hat{v}_{z_t} = {} & \frac{-GM\hat{z}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{3/2}}dt + \\
& \Bigg(P_{x_tx_t}\frac{GM\hat{z}_t(\widehat{x^2}_t+\widehat{y^2}_t-4\widehat{z^2}_t)}{2(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} - P_{x_ty_t}\frac{15GM\hat{x}_t\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} \\
& -P_{x_tz_t}\frac{3GM\hat{x}_t(4\widehat{z^2}_t-\widehat{x^2}_t-\widehat{y^2}_t)}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} + P_{y_ty_t}\frac{GM\hat{z}_t(\widehat{x^2}_t+\widehat{z^2}_t-4\widehat{y^2}_t)}{2(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} \\
& -P_{y_tz_t}\frac{3GM\hat{y}_t(4\widehat{z^2}_t-\widehat{x^2}_t-\widehat{y^2}_t)}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}} + P_{z_tz_t}\frac{GM\hat{z}_t(9\widehat{x^2}_t+9\widehat{y^2}_t-7\widehat{z^2}_t)}{2(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{7/2}}\Bigg)dt
\end{aligned}
\tag{25}
$$

and the variance evolutions are

$$dP_{x_tx_t} = 2P_{x_tv_{x_t}}dt \tag{26}$$

$$dP_{x_ty_t} = dP_{y_tx_t} = (P_{x_tv_{y_t}} + P_{y_tv_{x_t}})dt \tag{27}$$

$$dP_{x_tz_t} = dP_{z_tx_t} = (P_{x_tv_{z_t}} + P_{z_tv_{x_t}})dt \tag{28}$$

$$
\begin{aligned}
dP_{x_tv_{x_t}} = {} & dP_{v_{x_t}x_t} = \Bigg(- P_{x_tx_t}\frac{GM(\widehat{y^2}_t+\widehat{z^2}_t-2\widehat{x^2}_t)}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{5/2}} \\
& +P_{x_ty_t}\frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{5/2}} + P_{x_tz_t}\frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{5/2}} + P_{v_{x_t}v_{x_t}}\Bigg)dt
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
dP_{x_tv_{y_t}} = {} & dP_{v_{y_t}x_t} = \Bigg(P_{x_tx_t}\frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{5/2}} \\
& -P_{x_ty_t}\frac{GM(\widehat{x^2}_t+\widehat{z^2}_t-2\widehat{y^2}_t)}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{5/2}} + P_{x_tz_t}\frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t+\widehat{y^2}_t+\widehat{z^2}_t)^{5/2}} + P_{v_{y_t}v_{x_t}}\Bigg)dt
\end{aligned}
\tag{30}
$$

$$dP_{x_t v_{z_t}} = dP_{v_{z_t} x_t} = \Bigg(P_{x_t x_t} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \tag{31}$$

$$+ P_{x_t y_t} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{x_t z_t} \frac{GM(\widehat{x^2}_t + \widehat{y^2}_t - 2\widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{z_t} v_{x_t}} \Bigg) dt$$

$$dP_{y_t y_t} = 2P_{y_t v_{y_t}} dt \tag{32}$$

$$dP_{y_t z_t} = dP_{z_t y_t} = (P_{y_t v_{z_t}} + P_{z_t v_{y_t}}) dt \tag{33}$$

$$dP_{y_t v_{x_t}} = dP_{v_{x_t} y_t} = \Bigg(- P_{x_t y_t} \frac{GM(\widehat{y^2}_t + \widehat{z^2}_t - 2\widehat{x^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \tag{34}$$

$$+ P_{y_t y_t} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{y_t z_t} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{x_t} v_{y_t}} \Bigg) dt$$

$$dP_{y_t v_{y_t}} = dP_{v_{y_t} y_t} = \Bigg(P_{x_t y_t} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \tag{35}$$

$$- P_{y_t y_t} \frac{GM(\widehat{x^2}_t + \widehat{z^2}_t - 2\widehat{y^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{y_t z_t} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{y_t} v_{y_t}} \Bigg) dt$$

$$dP_{y_t v_{z_t}} = dP_{v_{z_t} y_t} = \Bigg(P_{x_t y_t} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \tag{36}$$

$$+ P_{y_t y_t} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{y_t z_t} \frac{GM(\widehat{x^2}_t + \widehat{y^2}_t - 2\widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{y_t} v_{z_t}} \Bigg) dt$$

$$dP_{z_t z_t} = 2P_{z_t v_{z_t}} dt \tag{37}$$

$$dP_{z_t v_{x_t}} = dP_{v_{x_t} z_t} = \Bigg(- P_{x_t z_t} \frac{GM(\widehat{y^2}_t + \widehat{z^2}_t - 2\widehat{x^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \tag{38}$$

$$+ P_{y_t z_t} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{z_t z_t} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{x_t} v_{z_t}} \Bigg) dt$$

$$
\begin{aligned}
dP_{z_t v_{y_t}} &= dP_{v_{y_t} z_t} = \Bigg(P_{x_t z_t} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&\quad - P_{y_t z_t} \frac{GM(\widehat{x^2}_t + \widehat{z^2}_t - 2\widehat{y^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{z_t z_t} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{y_t} v_{z_t}} \Bigg) dt
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
dP_{z_t v_{z_t}} &= dP_{v_{z_t} z_t} = \Bigg(P_{x_t z_t} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&\quad + P_{y_t z_t} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{z_t z_t} \frac{GM(\widehat{x^2}_t + \widehat{y^2}_t - 2\widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{v_{z_t} v_{z_t}} \Bigg) dt
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
dP_{v_{x_t} v_{x_t}} &= 2\Bigg(- P_{x_t v_{x_t}} \frac{GM(\widehat{y^2}_t + \widehat{z^2}_t - 2\widehat{x^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&\quad + P_{y_t v_{x_t}} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{z_t v_{x_t}} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + \sigma^2_{v_{x_t}} \Bigg) dt
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
dP_{v_{x_t} v_{y_t}} &= dP_{v_{y_t} v_{x_t}} = \Bigg(\bigg(P_{x_t v_{x_t}} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&\quad - P_{y_t v_{x_t}} \frac{GM(\widehat{x^2}_t + \widehat{z^2}_t - 2\widehat{y^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{z_t v_{x_t}} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \bigg) \\
&\quad + \bigg(- P_{x_t v_{y_t}} \frac{GM(\widehat{y^2}_t + \widehat{z^2}_t - 2\widehat{x^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{y_t v_{y_t}} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&\quad + P_{z_t v_{y_t}} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \bigg) \Bigg) dt
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
dP_{v_{x_t}v_{z_t}} &= dP_{v_{z_t}v_{x_t}} = \Bigg(\Bigg(P_{x_t v_{x_t}} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&+P_{y_t v_{x_t}} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{z_t v_{x_t}} \frac{GM(\widehat{x^2}_t + \widehat{y^2}_t - 2\widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}}\Bigg) \\
&+\Bigg(- P_{x_t v_{z_t}} \frac{GM(\widehat{y^2}_t + \widehat{z^2}_t - 2\widehat{x^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{y_t v_{z_t}} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&+P_{z_t v_{z_t}} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}}\Bigg)\Bigg)dt
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
dP_{v_{y_t}v_{y_t}} &= 2\Bigg(P_{x_t v_{y_t}} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&-P_{y_t v_{y_t}} \frac{GM(\widehat{x^2}_t + \widehat{z^2}_t - 2\widehat{y^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + P_{z_t v_{y_t}} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + \sigma^2_{v_{y_t}}\Bigg)dt
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
dP_{v_{y_t}v_{z_t}} &= dP_{v_{z_t}v_{y_t}} = \Bigg(\Bigg(P_{x_t v_{y_t}} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&+P_{y_t v_{y_t}} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{z_t v_{y_t}} \frac{GM(\widehat{x^2}_t + \widehat{y^2}_t - 2\widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}}\Bigg) \\
&+\Bigg(P_{x_t v_{z_t}} \frac{3GM\hat{x}_t\hat{y}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{y_t v_{z_t}} \frac{GM(\widehat{x^2}_t + \widehat{z^2}_t - 2\widehat{y^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&+P_{z_t v_{z_t}} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}}\Bigg)\Bigg)dt
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
dP_{v_{z_t}v_{z_t}} &= 2\Bigg(P_{x_t v_{z_t}} \frac{3GM\hat{x}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} \\
&+P_{y_t v_{z_t}} \frac{3GM\hat{y}_t\hat{z}_t}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} - P_{z_t v_{z_t}} \frac{GM(\widehat{x^2}_t + \widehat{y^2}_t - 2\widehat{z^2}_t)}{(\widehat{x^2}_t + \widehat{y^2}_t + \widehat{z^2}_t)^{5/2}} + \sigma^2_{v_{z_t}}\Bigg)dt
\end{aligned}
\tag{46}
$$

Note that the evolutions of the mean and variance, i.e. equations from (20)-(46) involve the first-order and second-order partial of the system non-linearity.

3.2 At the observation

First, we state the mean and variance evolutions for continuous-discrete filter for scalar case and subsequently, the evolutions are extended to vector case. The evolutions are given by ([1], p. 363)

$$\hat{\xi}_{t_k}^{t_k} = \hat{\xi}_{t_k}^{t_{k-1}} + (\widehat{\xi h} - \hat{\xi}\hat{h})[\widehat{(h-\hat{h})^2} + R]^{-1}(\zeta_{t_k} - \hat{\zeta}_{t_k}^{t_{k-1}}) \tag{47}$$

$$P_{t_k}^{t_k} = P_{t_k}^{t_{k-1}} - (\widehat{\xi h} - \hat{\xi}\hat{h})[\widehat{(h-\hat{h})^2} + R]^{-1}(\widehat{h\xi} - \hat{h}\hat{\xi}) \tag{48}$$

The above equations in matrix-vector format can be expressed as

$$\hat{\xi}_{t_k}^{t_k} = \hat{\xi}_{t_k}^{t_{k-1}} + (\widehat{\xi h^T} - \hat{\xi}\hat{h}^T)[\widehat{\big((h-\hat{h})(h-\hat{h})^T\big)} + R]^{-1}(\zeta_{t_k} - \hat{\zeta}_{t_k}^{t_{k-1}}) \tag{49}$$

$$P_{t_k}^{t_k} = P_{t_k}^{t_{k-1}} - (\widehat{\xi h^T} - \hat{\xi}\hat{h}^T)[\widehat{\big((h-\hat{h})(h-\hat{h})^T\big)} + R]^{-1}(\widehat{h\xi^T} - \hat{h}\hat{\xi}^T) \tag{50}$$

For numerical simulations, the component version is more attractive and convenient form. In component version, above equations become

$$(\hat{\xi}_{t_k}^{t_k})_i = (\hat{\xi}_{t_k}^{t_{k-1}})_i + (\widehat{\xi_i h} - \hat{\xi}_i\hat{h})^T[\widehat{\big((h-\hat{h})(h-\hat{h})^T\big)} + R]^{-1}(\zeta_{t_k} - \hat{\zeta}_{t_k}^{t_{k-1}}) \tag{51}$$

$$(P_{t_k}^{t_k})_{ij} = (P_{t_k}^{t_{k-1}})_{ij} - (\widehat{\xi_i h} - \hat{\xi}_i\hat{h})^T[\widehat{\big((h-\hat{h})(h-\hat{h})^T\big)} + R]^{-1}(\widehat{h\xi_j} - \hat{h}\hat{\xi}_j) \tag{52}$$

After second-order approximation, we get

$$\begin{aligned}(\hat{\xi}_{t_k}^{t_k})_i &= (\hat{\xi}_{t_k}^{t_{k-1}})_i + \left(\sum_k P_{ik}\frac{\partial h^T(\hat{\xi},t_k)}{\partial\xi_k}\right)\Bigg(\sum_{p,q} P_{pq}\frac{\partial h(\hat{\xi},t_k)}{\partial\xi_p}\frac{\partial h^T(\hat{\xi},t_k)}{\partial\xi_q} \\ &\quad + R - \frac{1}{4}\sum_{j,k}\sum_{l,m} P_{jk}P_{lm}\frac{\partial^2 h(\hat{\xi},t_k)}{\partial\xi_j\partial\xi_k}\frac{\partial^2 h^T(\hat{\xi},t_k)}{\partial\xi_l\partial\xi_m}\Bigg)^{-1} \\ &\quad \times\left(\zeta_{t_k} - h(\hat{\xi},t_k) - \frac{1}{2}\sum_{jk} p_{jk}\frac{\partial^2 h(\hat{\xi},t_k)}{\partial\xi_j\partial\xi_k}\right)\end{aligned} \tag{53}$$

$$\begin{aligned}(P_{t_k}^{t_k})_{ij} &= (P_{t_k}^{t_{k-1}})_{ij} - \left(\sum_k P_{ik}\frac{\partial h^T(\hat{\xi},t_k)}{\partial\xi_k}\right)\Bigg(\sum_{p,q} P_{pq}\frac{\partial h(\hat{\xi},t_k)}{\partial\xi_p}\frac{\partial h^T(\hat{\xi},t_k)}{\partial\xi_q} \\ &\quad + R - \frac{1}{4}\sum_{j,k}\sum_{l,m} P_{jk}P_{lm}\frac{\partial^2 h(\hat{\xi},t_k)}{\partial\xi_j\partial\xi_k}\frac{\partial^2 h^T(\hat{\xi},t_k)}{\partial\xi_l\partial\xi_m}\Bigg)^{-1} \\ &\quad \times\left(\sum_k P_{kj}\frac{\partial h(\hat{\xi},t_k)}{\partial\xi_k}\right)\end{aligned} \tag{54}$$

Let

$$\hat{r}_{t_k}^{t_{k-1}} = \sqrt{\widehat{x^2}_{t_k}^{t_{k-1}} + \widehat{y^2}_{t_k}^{t_{k-1}} + \widehat{z^2}_{t_k}^{t_{k-1}}} \tag{55}$$

and

$$\hat{r}_{t_k}^{2_{t_{k-1}}} = \widehat{x^2}_{t_k}^{t_{k-1}} + \widehat{y^2}_{t_k}^{t_{k-1}} + \widehat{z^2}_{t_k}^{t_{k-1}} \tag{56}$$

$$\hat{E}_{t_k}^{t_{k-1}} = \arctan\left(\frac{\hat{z}_{t_k}^{t_{k-1}}}{\sqrt{\widehat{x^2}_{t_k}^{t_{k-1}} + \widehat{y^2}_{t_k}^{t_{k-1}}}}\right) \tag{57}$$

$$\hat{A}_{t_k}^{t_{k-1}} = \arctan\left(\frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{y}_{t_k}^{t_{k-1}}}\right) \tag{58}$$

$$\hat{\rho}_{t_k}^{t_{k-1}} = \sqrt{\widehat{x^2}_{t_k}^{t_{k-1}} + \widehat{y^2}_{t_k}^{t_{k-1}}} \tag{59}$$

$$\hat{\rho}_{t_k}^{2_{t_{k-1}}} = \widehat{x^2}_{t_k}^{t_{k-1}} + \widehat{y^2}_{t_k}^{t_{k-1}} \tag{60}$$

Making the use of equations from (6)-(8) and (53)-(60), the mean evolutions are

$$(\hat{\xi}_{t_k}^{t_k})_i = (\hat{\xi}_{t_k}^{t_{k-1}})_i + HYZ \tag{61}$$

where

$$H = \left(\sum_k P_{ik}\frac{\partial h^T(\hat{\xi}, t_k)}{\partial \xi_k}\right) \quad (1 \times 3)$$

$$\begin{aligned} H &= \begin{pmatrix} H_{11} & H_{12} & H_{13} \end{pmatrix} \\ &= \begin{pmatrix} \sum_k P_{ik}\frac{\partial h_1(\hat{\xi}, t_k)}{\partial \xi_k} & \sum_k P_{ik}\frac{\partial h_2(\hat{\xi}, t_k)}{\partial \xi_k} & \sum_k P_{ik}\frac{\partial h_3(\hat{\xi}, t_k)}{\partial \xi_k} \end{pmatrix} \end{aligned}$$

$$H_{11} = P_{ix_{t_k}}^{t_{k-1}}\frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}} + P_{iy_{t_k}}^{t_{k-1}}\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}} + P_{iz_{t_k}}^{t_{k-1}}\frac{\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}} \tag{62}$$

$$H_{12} = P_{ix_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}} + P_{iy_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}} + P_{iz_{t_k}}^{t_{k-1}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}} \tag{63}$$

$$H_{13} = P^{t_{k-1}}_{ix_{t_k}} \frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} - P^{t_{k-1}}_{iy_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} \tag{64}$$

and

$$\begin{aligned} Y &= \left(\sum_{p,q} P_{pq} \frac{\partial h(\hat{\xi}, t_k)}{\partial \xi_p} \frac{\partial h^T(\hat{\xi}, t_k)}{\partial \xi_q} \right. \\ &\quad \left. + R - \frac{1}{4} \sum_{j,k} \sum_{l,m} P_{jk} P_{lm} \frac{\partial^2 h(\hat{\xi}, t_k)}{\partial \xi_j \partial \xi_k} \frac{\partial^2 h^T(\hat{\xi}, t_k)}{\partial \xi_l \partial \xi_m} \right)^{-1} \quad (3 \times 3) \end{aligned}$$

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \\ & & \end{pmatrix}$$

$$\begin{aligned} Y_{11} &= \frac{1}{\Delta}(a_5 a_9 - a_6 a_8), Y_{12} = \frac{1}{\Delta}(-a_2 a_9 + a_3 a_8), \\ Y_{13} &= \frac{1}{\Delta}(a_2 a_6 - a_3 a_5), Y_{21} = \frac{1}{\Delta}(-a_4 a_9 + a_6 a_7) \end{aligned}$$

$$\begin{aligned} Y_{22} &= \frac{1}{\Delta}(a_1 a_9 - a_3 a_7), Y_{23} = \frac{1}{\Delta}(-a_1 a_6 + a_3 a_4), \\ Y_{31} &= \frac{1}{\Delta}(a_4 a_8 - a_5 a_7), Y_{32} = \frac{1}{\Delta}(-a_1 a_8 + a_2 a_7) \end{aligned}$$

$$\begin{aligned} Y_{33} &= \frac{1}{\Delta}(-a_1 a_5 + a_2 a_4), \\ \Delta &= a_1 a_5 a_9 - a_1 a_6 a_8 - a_4 a_2 a_9 + a_4 a_3 a_8 + a_7 a_2 a_6 - a_7 a_3 a_5 \end{aligned}$$

$$
\begin{aligned}
a_1 \quad = \quad & \frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\right) \qquad (65)\\
& +\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\left(P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\right)\\
& +\frac{\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\left(P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\right)\\
& +\sigma_R^2-\frac{1}{4}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\widehat{y^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}-2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\right.\\
& -2P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\widehat{x^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\\
& \left.+2P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{-\hat{y}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{\widehat{x^2}_{t_k}^{t_{k-1}}+\widehat{y^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
a_2 \quad = \quad & \frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\right) \qquad (66)\\
& +\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\left(P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\right)\\
& +\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\left(P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\right)\\
& -\frac{1}{4}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\widehat{y^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}-2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}-2P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\right.\\
& \left.+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\widehat{x^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}+2P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{-\hat{y}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{\widehat{x^2}_{t_k}^{t_{k-1}}+\widehat{y^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\right)\\
& \times\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\hat{z}_{t_k}^{t_{k-1}}\widehat{y^2}_{t_k}^{t_{k-1}}(\widehat{x^2}_{t_k}^{t_{k-1}}-\widehat{y^2}_{t_k}^{t_{k-1}}-\widehat{z^2}_{t_k}^{t_{k-1}})+2\hat{z}_{t_k}^{t_{k-1}}\widehat{x^4}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{3/2_{t_{k-1}}}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}\hat{z}^{t_{k-1}}_{t_k}\frac{(3\widehat{x^2}^{t_{k-1}}_{t_k}+3\widehat{y^2}^{t_{k-1}}_{t_k}+\widehat{z^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}}\\
&+2P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}\\
&+2P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}\\
&+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\hat{z}^{t_{k-1}}_{t_k}\widehat{x^2}^{t_{k-1}}_{t_k}(\widehat{y^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{z^2}^{t_{k-1}}_{t_k})+2\hat{z}^{t_{k-1}}_{t_k}\widehat{y^4}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}}\\
&\left.+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{-2\hat{z}^{t_{k-1}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
a_3 \;=\;& \frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}-P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}\right)\\
&+\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}-P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}\right)\\
&+\frac{\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}-P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}\right)\\
&-\frac{1}{4}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{-2\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}}+2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{2\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}}\right)\\
&\times\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\widehat{y^2}^{t_{k-1}}_{t_k}+\widehat{z^2}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}}-2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}}-2P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}}\right.\\
&\left.+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\widehat{x^2}^{t_{k-1}}_{t_k}+\widehat{z^2}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}}+2P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{-\hat{y}^{t_{k-1}}_{t_k}\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}}\right)
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
a_4 \;=\;& \frac{-\hat{z}^{t_{k-1}}_{t_k}\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}+P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}+P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}\right)\\
&+\frac{-\hat{z}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}+P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}}\right)
\end{aligned}
\tag{68}
$$

$$+\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\left(P_{x_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P_{y_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}+P_{z_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{t_{k-1}}}\right)$$

$$-\frac{1}{4}\left(P_{x_{t_k}x_{t_k}}^{t_{k-1}}\frac{\widehat{y^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}-2P_{x_{t_k}y_{t_k}}^{t_{k-1}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}-2P_{x_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\right.$$

$$\left.+P_{y_{t_k}y_{t_k}}^{t_{k-1}}\frac{\widehat{x^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}-2P_{y_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{y}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}+P_{z_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{\rho}_{t_k}^{2_{t_{k-1}}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}}\right)$$

$$\times\left(P_{x_{t_k}x_{t_k}}^{t_{k-1}}\frac{\hat{z}_{t_k}^{t_{k-1}}\widehat{y^2}_{t_k}^{t_{k-1}}(\widehat{x^2}_{t_k}^{t_{k-1}}-\widehat{y^2}_{t_k}^{t_{k-1}}-\widehat{z^2}_{t_k}^{t_{k-1}})+2\hat{z}_{t_k}^{t_{k-1}}\widehat{x^4}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{3/2_{t_{k-1}}}}\right.$$

$$+2P_{x_{t_k}y_{t_k}}^{t_{k-1}}\frac{\hat{x}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}\hat{z}_{t_k}^{t_{k-1}}(3\widehat{x^2}_{t_k}^{t_{k-1}}+3\widehat{y^2}_{t_k}^{t_{k-1}}+\widehat{z^2}_{t_k}^{t_{k-1}})}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{3/2_{t_{k-1}}}}+$$

$$2P_{x_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{x}_{t_k}^{t_{k-1}}(\widehat{z^2}_{t_k}^{t_{k-1}}-\widehat{x^2}_{t_k}^{t_{k-1}}-\widehat{y^2}_{t_k}^{t_{k-1}})}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}$$

$$+P_{y_{t_k}y_{t_k}}^{t_{k-1}}\frac{\hat{z}_{t_k}^{t_{k-1}}\widehat{x^2}_{t_k}^{t_{k-1}}(\widehat{y^2}_{t_k}^{t_{k-1}}-\widehat{x^2}_{t_k}^{t_{k-1}}-\widehat{z^2}_{t_k}^{t_{k-1}})+2\hat{z}_{t_k}^{t_{k-1}}\widehat{y^4}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{3/2_{t_{k-1}}}}$$

$$\left.+2P_{y_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{y}_{t_k}^{t_{k-1}}(\widehat{z^2}_{t_k}^{t_{k-1}}-\widehat{x^2}_{t_k}^{t_{k-1}}-\widehat{y^2}_{t_k}^{t_{k-1}})}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}-P_{z_{t_k}z_{t_k}}^{t_{k-1}}\frac{2\hat{z}_{t_k}^{t_{k-1}}\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}}}\right)$$

$$a_5 = \frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}\left(P_{x_{t_k}x_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P_{x_{t_k}y_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P_{x_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\right)$$

$$+\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}\left(P_{x_{t_k}y_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P_{y_{t_k}y_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P_{y_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\right)$$

$$+\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\left(P_{x_{t_k}z_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{x}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P_{y_{t_k}z_{t_k}}^{t_{k-1}}\frac{-\hat{z}_{t_k}^{t_{k-1}}\hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}\hat{\rho}_{t_k}^{t_{k-1}}}+P_{z_{t_k}z_{t_k}}^{t_{k-1}}\frac{\hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{2_{t_{k-1}}}}\right)+$$

$$\sigma_E^2-\frac{1}{4}\left(P_{x_{t_k}x_{t_k}}^{t_{k-1}}\frac{\hat{z}_{t_k}^{t_{k-1}}\widehat{y^2}_{t_k}^{t_{k-1}}(\widehat{x^2}_{t_k}^{t_{k-1}}-\widehat{y^2}_{t_k}^{t_{k-1}}-\widehat{z^2}_{t_k}^{t_{k-1}})+2\hat{z}_{t_k}^{t_{k-1}}\widehat{x^4}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}}\hat{\rho}_{t_k}^{3/2_{t_{k-1}}}}\right.$$

$$
\begin{aligned}
&+2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}\hat{z}^{t_{k-1}}_{t_k}(3\widehat{x^2}^{t_{k-1}}_{t_k}+3\widehat{y^2}^{t_{k-1}}_{t_k}+\widehat{z^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
&+2P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}} \\
&+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\hat{z}^{t_{k-1}}_{t_k}\widehat{x^2}^{t_{k-1}}_{t_k}(\widehat{y^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{z^2}^{t_{k-1}}_{t_k})+2\hat{z}^{t_{k-1}}_{t_k}\widehat{y^4}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
&+2P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{-2\hat{z}^{t_{k-1}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}}\Bigg)^2
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
a_6 \;=\; &\frac{-\hat{z}^{t_{k-1}}_{t_k}\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}+P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{-\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}\right) \\
&+\frac{-\hat{z}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{-\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}\right) \\
&+\frac{\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}}\left(P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}+P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{-\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}\right) \\
&-\frac{1}{4}\left(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{-2\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}}+2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}}+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{2\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}}\right) \\
&\times\Bigg(P^{t_{k-1}}_{x_{t_k}x_{t_k}}\frac{\hat{z}^{t_{k-1}}_{t_k}\widehat{y^2}^{t_{k-1}}_{t_k}(\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k}-\widehat{z^2}^{t_{k-1}}_{t_k})+2\hat{z}^{t_{k-1}}_{t_k}\widehat{x^4}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
&+2P^{t_{k-1}}_{x_{t_k}y_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}\hat{z}^{t_{k-1}}_{t_k}(3\widehat{x^2}^{t_{k-1}}_{t_k}+3\widehat{y^2}^{t_{k-1}}_{t_k}+\widehat{z^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
&+2P^{t_{k-1}}_{x_{t_k}z_{t_k}}\frac{\hat{x}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}} \\
&+P^{t_{k-1}}_{y_{t_k}y_{t_k}}\frac{\hat{z}^{t_{k-1}}_{t_k}\widehat{x^2}^{t_{k-1}}_{t_k}(\widehat{y^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{z^2}^{t_{k-1}}_{t_k})+2\hat{z}^{t_{k-1}}_{t_k}\widehat{y^4}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
&+2P^{t_{k-1}}_{y_{t_k}z_{t_k}}\frac{\hat{y}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k}-\widehat{x^2}^{t_{k-1}}_{t_k}-\widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}+P^{t_{k-1}}_{z_{t_k}z_{t_k}}\frac{-2\hat{z}^{t_{k-1}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}}\Bigg)
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
a_7 \quad = \quad & \frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} \left(P^{t_{k-1}}_{x_{t_k} x_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{x_{t_k} y_{t_k}} \frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} + P_{x_{t_k} z^{t_{k-1}}_{t_k}} \frac{\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} \right) \qquad (71) \\
& + \frac{-\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} \left(P_{x_{t_k} y_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{y_{t_k} y_{t_k}} \frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{y_{t_k} z_{t_k}} \frac{\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} \right) \\
& - \frac{1}{4} \left(P^{t_{k-1}}_{x_{t_k} x_{t_k}} \frac{-2\hat{x}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} + 2P^{t_{k-1}}_{x_{t_k} y_{t_k}} \frac{{\widehat{x^2}}^{t_{k-1}}_{t_k} - {\widehat{y^2}}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} \right. \\
& \left. + P^{t_{k-1}}_{y_{t_k} y_{t_k}} \frac{2\hat{x}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} \right) \left(P^{t_{k-1}}_{x_{t_k} x_{t_k}} \frac{{\widehat{y^2}}^{t_{k-1}}_{t_k} + {\widehat{z^2}}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2)t_{k-1}}_{t_k}} \right. \\
& - 2P^{t_{k-1}}_{x_{t_k} y_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}} - 2P^{t_{k-1}}_{x_{t_k} z_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k} \hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}} + P^{t_{k-1}}_{y_{t_k} y_{t_k}} \frac{{\widehat{x^2}}^{t_{k-1}}_{t_k} + {\widehat{z^2}}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}} \\
& \left. + 2P^{t_{k-1}}_{y_{t_k} z_{t_k}} \frac{-\hat{y}^{t_{k-1}}_{t_k} \hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}} + P^{t_{k-1}}_{z_{t_k} z_{t_k}} \frac{\hat{\rho}^{2_{t_{k-1}}}_{t_k}}{\hat{r}^{3/2_{t_{k-1}}}_{t_k}} \right)
\end{aligned}
$$

$$
\begin{aligned}
a_8 \quad = \quad & \frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} \left(P^{t_{k-1}}_{x_{t_k} x_{t_k}} \frac{-\hat{z}^{t_{k-1}}_{t_k} \hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k} \hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{x_{t_k} y_{t_k}} \frac{-\hat{z}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k} \hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{x_{t_k} z_{t_k}} \frac{\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}} \right) \qquad (72) \\
& + \frac{-\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} \left(P^{t_{k-1}}_{x_{t_k} y_{t-k}} \frac{-\hat{z}^{t_{k-1}}_{t_k} \hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k} \hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{y_{t_k} y_{t_k}} \frac{-\hat{z}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k} \hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{y_{t_k} z_{t_k}} \frac{\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}} \right) \\
& - \frac{1}{4} \left(P^{t_{k-1}}_{x_{t_k} x_{t_k}} \frac{-2\hat{x}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} + 2P^{t_{k-1}}_{x_{t_k} y_{t_k}} \frac{{\widehat{x^2}}^{t_{k-1}}_{t_k} - {\widehat{y^2}}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} + P^{t_{k-1}}_{y_{t_k} y_{t_k}} \frac{2\hat{x}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} \right) \\
& \left(P^{t_{k-1}}_{x_{t_k} x_{t_k}} \frac{\hat{z}^{t_{k-1}}_{t_k} {\widehat{y^2}}^{t_{k-1}}_{t_k} ({\widehat{x^2}}^{t_{k-1}}_{t_k} - {\widehat{y^2}}^{t_{k-1}}_{t_k} - {\widehat{z^2}}^{t_{k-1}}_{t_k}) + 2\hat{z}^{t_{k-1}}_{t_k} {\widehat{x^4}}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k} \hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \right. \\
& + 2P^{t_{k-1}}_{x_{t_k} y_{t_k}} \hat{x}^{t_{k-1}}_{t_k} \hat{y}^{t_{k-1}}_{t_k} \hat{z}^{t_{k-1}}_{t_k} \frac{(3{\widehat{x^2}}^{t_{k-1}}_{t_k} + 3{\widehat{y^2}}^{t_{k-1}}_{t_k} + {\widehat{z^2}}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k} \hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
& + 2P_{x_{t_k} z_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k} ({\widehat{z^2}}^{t_{k-1}}_{t_k} - {\widehat{x^2}}^{t_{k-1}}_{t_k} - {\widehat{y^2}}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k} \hat{\rho}^{t_{k-1}}_{t_k}} \\
& + 2P^{t_{k-1}}_{y_{t_k} z_{t_k}} \frac{\hat{y}^{t_{k-1}}_{t_k} ({\widehat{z^2}}^{t_{k-1}}_{t_k} - {\widehat{x^2}}^{t_{k-1}}_{t_k} - {\widehat{y^2}}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k} \hat{\rho}^{t_{k-1}}_{t_k}}
\end{aligned}
$$

$$+P_{y_{t_k}y_{t_k}}^{t_{k-1}} \frac{\hat{z}_{t_k}^{t_{k-1}} \widehat{x^2}_{t_k}^{t_{k-1}} (\widehat{y^2}_{t_k}^{t_{k-1}} - \widehat{x^2}_{t_k}^{t_{k-1}} - \widehat{z^2}_{t_k}^{t_{k-1}}) + 2\hat{z}_{t_k}^{t_{k-1}} \widehat{y^4}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}} \hat{\rho}_{t_k}^{3/2_{t_{k-1}}}}$$

$$+P_{z_{t_k}z_{t_k}}^{t_{k-1}} \frac{-2\hat{z}_{t_k}^{t_{k-1}} \hat{\rho}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{4_{t_{k-1}}}} \Bigg)$$

$$\begin{aligned} a_9 \quad = \quad & \frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{2_{t_{k-1}}}} \left(P_{x_{t_k}x_{t_k}} \frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{2_{t_{k-1}}}} + P_{x_{t_k}y_{t_k}}^{t_{k-1}} \frac{-\hat{x}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{2_{t_{k-1}}}} \right) \\ & - \frac{\hat{x}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{2_{t_{k-1}}}} \left(P_{x_{t_k}y_{t_k}}^{t_{k-1}} \frac{\hat{y}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{2_{t_{k-1}}}} + P_{y_{t_k}y_{t_k}}^{t_{k-1}} \frac{-\hat{x}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{2_{t_{k-1}}}} \right) + \frac{\sigma_E^2}{cos^2(E)} \\ & - \frac{1}{4} \left(P_{x_{t_k}x_{t_k}}^{t_{k-1}} \frac{-2\hat{x}_{t_k}^{t_{k-1}} \hat{y}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{4_{t_{k-1}}}} + 2P_{x_{t_k}y_{t_k}}^{t_{k-1}} \frac{\widehat{x^2}_{t_k}^{t_{k-1}} - \widehat{y^2}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{4_{t_{k-1}}}} + P_{y_{t_k}y_{t_k}}^{t_{k-1}} \frac{2\hat{x}_{t_k}^{t_{k-1}} \hat{y}_{t_k}^{t_{k-1}}}{\hat{\rho}_{t_k}^{4_{t_{k-1}}}} \right)^2 \end{aligned} \tag{73}$$

and

$$Z = \left(\zeta_{t_k} - h(\hat{\xi}, t_k) - \frac{1}{2} \sum_{jk} p_{jk} \frac{\partial^2 h(\hat{\xi}, t_k)}{\partial \xi_j \partial \xi_k} \right) \quad (3 \times 1)$$

$$Z = \begin{pmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{pmatrix} = \begin{pmatrix} \zeta_{1t_k} - h_1(\hat{\xi}, t_k) - \frac{1}{2} \sum_{jk} p_{jk} \frac{\partial^2 h_1(\hat{\xi}, t_k)}{\partial \xi_j \partial \xi_k} \\ \zeta_{2t_k} - h_2(\hat{\xi}, t_k) - \frac{1}{2} \sum_{jk} p_{jk} \frac{\partial^2 h_2(\hat{\xi}, t_k)}{\partial \xi_j \partial \xi_k} \\ \zeta_{3t_k} - h_3(\hat{\xi}, t_k) - \frac{1}{2} \sum_{jk} p_{jk} \frac{\partial^2 h_3(\hat{\xi}, t_k)}{\partial \xi_j \partial \xi_k} \end{pmatrix}$$

$$\begin{aligned} Z_{11} \quad = \quad & \zeta_{1t_k} - r_{t_k} - \left(P_{x_{t_k}x_{t_k}}^{t_{k-1}} \frac{\widehat{y^2}_{t_k}^{t_{k-1}} + \widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}} \right. \\ & -2P_{x_{t_k}y_{t_k}}^{t_{k-1}} \frac{\hat{x}_{t_k}^{t_{k-1}} \hat{y}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}} - 2P_{x_{t_k}z_{t_k}}^{t_{k-1}} \frac{\hat{x}_{t_k}^{t_{k-1}} \hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}} + P_{y_{t_k}y_{t_k}}^{t_{k-1}} \frac{\widehat{x^2}_{t_k}^{t_{k-1}} + \widehat{z^2}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}} \\ & \left. +2P_{y_{t_k}z_{t_k}}^{t_{k-1}} \frac{-\hat{y}_{t_k}^{t_{k-1}} \hat{z}_{t_k}^{t_{k-1}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}} + P_{z_{t_k}z_{t_k}}^{t_{k-1}} \frac{\hat{\rho}_{t_k}^{2_{t_{k-1}}}}{\hat{r}_{t_k}^{3/2_{t_{k-1}}}} \right) \end{aligned} \tag{74}$$

$$
\begin{aligned}
Z_{21} \;=\; & \zeta_{2t_k} - E_{t_k} \qquad (75)\\
& -\frac{1}{2}\Bigg(P^{t_{k-1}}_{x_{t_k}x_{t_k}} \frac{\hat{z}^{t_{k-1}}_{t_k}\widehat{y^2}^{t_{k-1}}_{t_k}(\widehat{x^2}^{t_{k-1}}_{t_k} - \widehat{y^2}^{t_{k-1}}_{t_k} - \widehat{z^2}^{t_{k-1}}_{t_k}) + 2\hat{z}^{t_{k-1}}_{t_k}\widehat{x^4}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
& +2P^{t_{k-1}}_{x_{t_k}y_{t_k}} \hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}\hat{z}^{t_{k-1}}_{t_k} \frac{(3\widehat{x^2}^{t_{k-1}}_{t_k} + 3\widehat{y^2}^{t_{k-1}}_{t_k} + \widehat{z^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
& +2P^{t_{k-1}}_{x_{t_k}z_{t_k}} \frac{\hat{x}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k} - \widehat{x^2}^{t_{k-1}}_{t_k} - \widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{z_{t_k}z_{t_k}} \frac{-2\hat{z}^{t_{k-1}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}} \\
& +P^{t_{k-1}}_{y_{t_k}y_{t_k}} \frac{\hat{z}^{t_{k-1}}_{t_k}\widehat{x^2}^{t_{k-1}}_{t_k}(\widehat{y^2}^{t_{k-1}}_{t_k} - \widehat{x^2}^{t_{k-1}}_{t_k} - \widehat{z^2}^{t_{k-1}}_{t_k}) + 2\hat{z}^{t_{k-1}}_{t_k}\widehat{y^4}^{t_{k-1}}_{t_k}}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{3/2_{t_{k-1}}}_{t_k}} \\
& +2P^{t_{k-1}}_{y_{t_k}z_{t_k}} \frac{\hat{y}^{t_{k-1}}_{t_k}(\widehat{z^2}^{t_{k-1}}_{t_k} - \widehat{x^2}^{t_{k-1}}_{t_k} - \widehat{y^2}^{t_{k-1}}_{t_k})}{\hat{r}^{4_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
Z_{31} \;=\; & \zeta_{2t_k} - A_{t_k} - \frac{1}{2}\Bigg(P^{t_{k-1}}_{x_{t_k}x_{t_k}} \frac{-2\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} \qquad (76)\\
& +2P^{t_{k-1}}_{x_{t_k}y_{t_k}} \frac{\widehat{x^2}^{t_{k-1}}_{t_k} - \widehat{y^2}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} + P^{t_{k-1}}_{y_{t_k}y_{t_k}} \frac{2\hat{x}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{4_{t_{k-1}}}_{t_k}} \Bigg)
\end{aligned}
$$

Finally, the mean in component form is

$$
\begin{aligned}
(\hat{\xi}^{t_k}_{t_k})_i \;=\; & (\hat{\xi}^{t_{k-1}}_{t_k})_i + H_{11}\big(Y_{11}Z_{11} + Y_{12}Z_{21} + Y_{13}Z_{31}\big) \qquad (77)\\
& +H_{12}\big(Y_{21}Z_{11} + Y_{22}Z_{21} + Y_{23}Z_{31}\big)\\
& +H_{13}\big(Y_{31}Z_{11} + Y_{32}Z_{21} + Y_{33}Z_{31}\big)
\end{aligned}
$$

and the variance evolutions are

$$
(P^{t_k}_{t_k})_{ij} = (P^{t_{k-1}}_{t_k})_{ij} - HYG \qquad (78)
$$

where

$$
G = \left(\sum_k P_{kj} \frac{\partial h(\hat{\xi}, t_k)}{\partial \xi_k} \right) \quad (3 \times 1)
$$

$$
G = \begin{pmatrix} G_{11} \\ G_{21} \\ G_{31} \end{pmatrix} = \begin{pmatrix} \sum_k P_{kj} \frac{\partial h_1(\hat{\xi}, t_k)}{\partial \xi_k} \\ \sum_k P_{kj} \frac{\partial h_2(\hat{\xi}, t_k)}{\partial \xi_k} \\ \sum_k P_{kj} \frac{\partial h_3(\hat{\xi}, t_k)}{\partial \xi_k} \end{pmatrix}
$$

$$G_{11} = P^{t_{k-1}}_{x_{t_k}j}\frac{\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{y_{t_k}j}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{z_{t_k}j}\frac{\hat{z}^{t_{k-1}}_{t_k}}{\hat{r}^{t_{k-1}}_{t_k}} \tag{79}$$

$$G_{21} = P^{t_{k-1}}_{x_{t_k}j}\frac{-\hat{z}^{t_{k-1}}_{t_k}\hat{x}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{y_{t_k}j}\frac{-\hat{z}^{t_{k-1}}_{t_k}\hat{y}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}\hat{\rho}^{t_{k-1}}_{t_k}} + P^{t_{k-1}}_{z_{t_k}j}\frac{\hat{\rho}^{t_{k-1}}_{t_k}}{\hat{r}^{2_{t_{k-1}}}_{t_k}} \tag{80}$$

$$G_{31} = P^{t_{k-1}}_{x_{t_k}j}\frac{\hat{y}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} + P^{t_{k-1}}_{y_{t_k}j}\frac{-\hat{x}^{t_{k-1}}_{t_k}}{\hat{\rho}^{2_{t_{k-1}}}_{t_k}} \tag{81}$$

$$\begin{aligned}(P^{t_k}_{t_k})_{ij} &= (P^{t_{k-1}}_{t_k})_{ij} - H_{11}\big(Y_{11}G_{11} + Y_{12}G_{21} + Y_{13}G_{31}\big) \\ &\quad + H_{12}\big(Y_{21}G_{11} + Y_{22}G_{21} + Y_{23}G_{31}\big) \\ &\quad + H_{13}\big(Y_{31}G_{11} + Y_{32}G_{21} + Y_{33}G_{31}\big)\end{aligned} \tag{82}$$

where

$$\hat{\xi}^{t_{k-1}}_{it_k} = \mathbb{E}\big(\xi_i(t_k)|\zeta_{t_{k-1}}\big)$$

and

$$P^{t_{k-1}}_{ij} = \mathbb{E}\Big(\big(\xi_{it_k} - \mathbb{E}(\xi_{it_k}|\zeta_{t_{k-1}})\big)\big(\xi_{jt_k} - \mathbb{E}(\xi_{jt_k}|\zeta_{t_{k-1}})\big)|\zeta_{t_{k-1}}\Big)$$

4 Simulation results

The standard discretization method for simulating an ordinary differential equation, by means of a difference equation is employed to simulate the mean and covariance evolution equations. The simplest way to simulate a differential or stochastic differential equation is to replace differentials by finite differences. Thus, to simulate stochastic differential equation $d\xi_t = f(\xi_t)dt + G(t)dB_t$ we would replace $d\xi_t$ by $(\xi_{t_{k+1}} - \xi_{t_k})$, dt by $(t_{k+1} - t_k)$ and dB_t by $\sqrt{t_{k+1} - t_k}W_{t_{k+1}}$, where $W_{t_{k+1}}$ is a standard normal variable. The numerical simulation of the estimation algorithm is based upon the discrete version of the stochastic dynamics and observation equation. The observations are generated using the equation $\zeta(t_k) = h(\zeta_{t_k}) + \nu(t_k)$. The measurement nonlinearity $h(\zeta(t_k))$ is generated from the stochastic dynamics and subsequently appropriate noise sequences were added to generate the measurements. Two different trajectories taken from [29] were simulated to examine the effectiveness of the second order continuous-discrete filter. Three sets of observations were used for each trajectory.

4.1 Trajectory I

The exact initial conditions for this trajectory are: $x_{t_0} = 1.06$ DU, $y_{t_0} = 0$, $z_{t_0} = 0$, $v_{x_{t_0}} = 0$, $v_{y_{t_0}} = -1.033$ DU/TU, $v_{z_{t_0}} = 0$. Strength of the stochastic acceleration $\sigma_x = \sigma_y = \sigma_z = 0.0012733 DU/TU^{3/2}$, and $t_k - t_{k-1} = 30$ seconds. Here t_k and t_{k-1} are observation instants.

4.2 Trajectory II

The initial conditions for this trajectory are $x_{t_0} = 1.06$ DU, $y_{t_0} = 0$, $z_{t_0} = 0$, $v_{x_{t_0}} = 0$, $v_{y_{t_0}} = 0.933$ DU/TU, $v_{z_{t_0}} = 0$. Strength of the stochastic acceleration $\sigma_x = \sigma_y = \sigma_z = 0.0012733 DU/TU^{3/2}$.

The initial covariance matrix being chosen as identically zero, which corresponds to the physical situation that initially the target is located at a definite point in the space and imparted a definite velocity. The initial conditions considered here are in 'canonical units' ([28] pp.40-41). In astrodynamics, A canonical unit is a unit of measurement defined in terms of an object's reference orbit. Canonical units are useful when the precise distances and masses of objects in space are not available. However, by setting the mass of a given object to be 1 mass unit and the mean distance of the reference object to another object in question, many calculations can be simplified. The distance unit DU is defined to be the mean radius of the reference orbit. The time unit TU is defined by the gravitational parameter $\mu = GM$. For canonical units, the gravitational parameter is defined as: $\mu = 1 \quad \frac{DU^3}{TU^2}$.

4.3 Measurement system

The measurement system consists of a phased array radar that measures the range and the direction cosines of the target. The rms noise in the measurements are taken as $10DU$ in range and $0.05DUrad$ in the direction cosines.

The numerical simulation of the nonlinear filters is intended not to make statistically correct evolutions, but to examine some of the qualitative characteristics of the filters in several computer runs [11]. Since, SNR is a popular performance measure of the Wiener filter, it can be regarded as 'effective performance measure' for the trajectory estimation problem [29]. If $\theta(t)$ is a parameter/physical quantity varies in a random or nonrandom function fashion over an interval of time $[0 \quad T]$, via an estimation, we obtain the estimate $\hat{\theta}(t)$ over the same duration, then the instantaneous error is $\theta(t) - \hat{\theta}(t)$

Table 1: SNR for Trajectory I & II

Observation noise variance	SNR for Trajectory I	SNR for Trajectory II
$\sigma_r = 10, \sigma_{angle} = 0.05$	9.5371	9.2161
$\sigma_r = 11, \sigma_{angle} = 0.08$	9.5371	9.2315
$\sigma_r = 14, \sigma_{angle} = 0.10$	9.5163	9.2857
$\sigma_r = 15, \sigma_{angle} = 0.50$	9.4898	9.2317

and SNR can be defined as the ratio of the mean square time cum ensemble averaged parameter over mean square cum ensemble averaged error process over the same interval.

$$SNR = \frac{\int_0^T \mathbb{E}\|\theta(t)\|^2 dt}{\int_o^T \mathbb{E}\|\theta(t) - \hat{\theta}(t)\|^2 dt}$$

we have applied this performance measure to the trajectory estimation problem. If SNR is somewhat closer to the order of ten, it reveals that the filter performance is fairly good and smaller than one, the filter is unable to eliminate the noise and poor. The proof of SNR(T) is given in [29]. Here, we simply state the empirical relation of the SNR(T) . The empirical SNR is defined as

$$SNR(T) = \frac{\sum_{k=0}^{n} \|\xi_{t_k}\|^2}{\sum_{k=0}^{n} \|\xi_{t_k} - \hat{\xi}_{t_k}^{t_k}\|^2} \tag{83}$$

4.4 Filtering results

Figures 3 and 4 summarize the results of first-order and second-order filters for trajectory I and Figures 5 and 6 - for trajectory II. Numerical simulations demonstrated in the figures, are aimed to illustrate the effect of the stochastic acceleration on the orbiting satellite via simulating conditional mean and variance evolutions of the problem considered here, i.e. equations (20)-(46), (77) and (82). Because of the second-order approximation of the system nonlinearity, 'Between the observations' the term GQG^T in the mean and variance evolution accounts for the stochastic perturbation felt by the orbiting satellite. The mean trajectory for the dust perturbed satellite using first-order approximation does not include variance term in mean evolution.

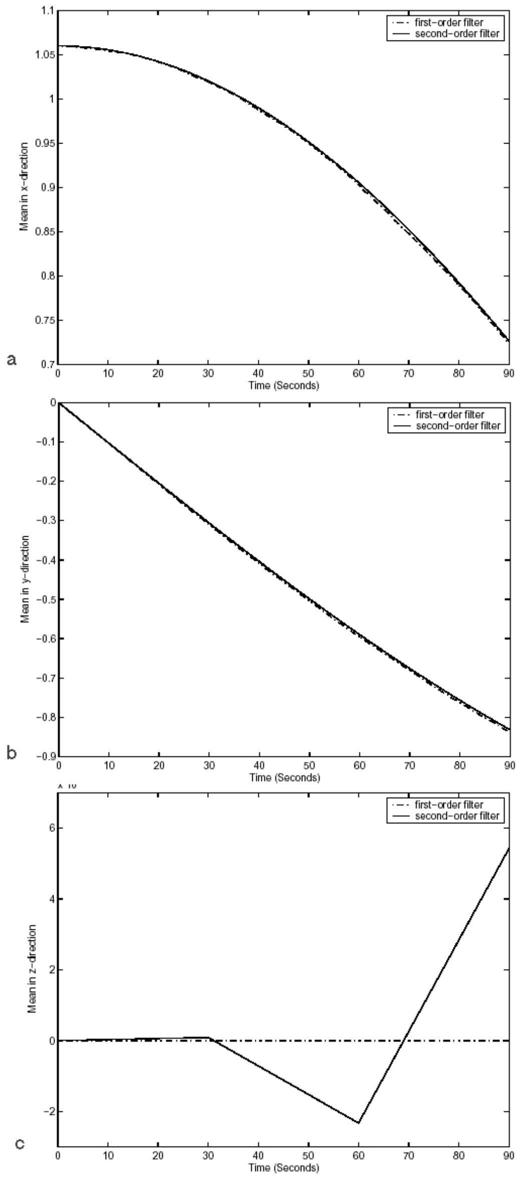

Figure 3: Trajectory I mean values: a) mean in x-direction; b) mean in y-direction; c) mean in z-direction.

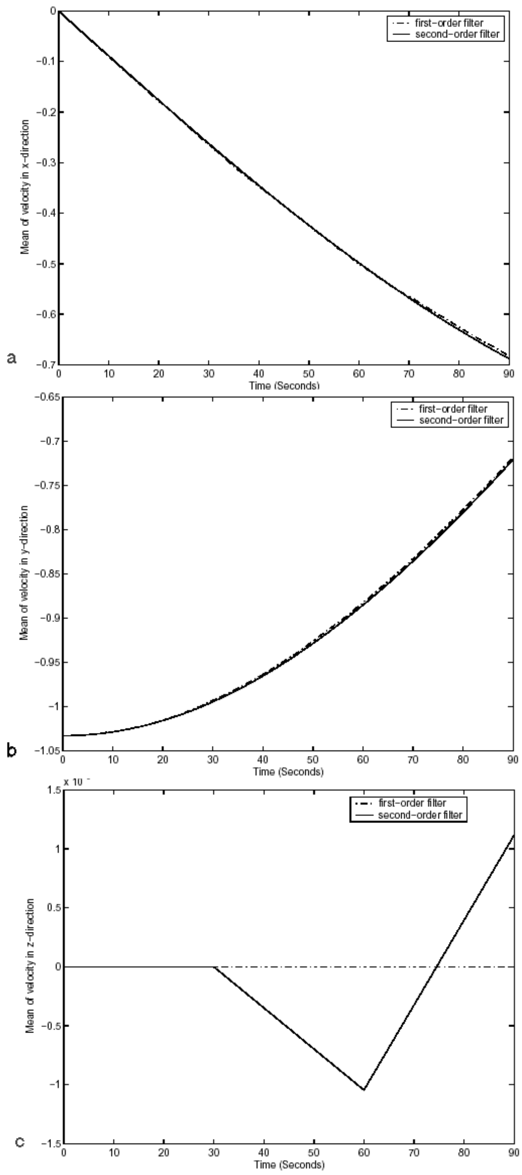

Figure 4: Trajectory I mean values of velocity: a) mean of velocity in x-direction; b) mean of velocity in y-direction; c) mean of velocity in z-direction.

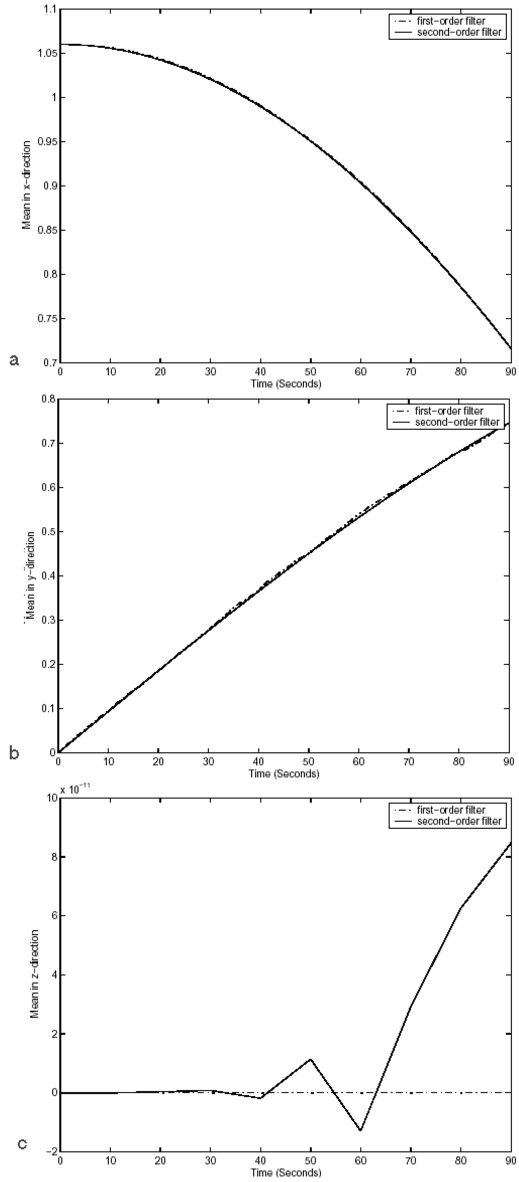

Figure 5: Trajectory II mean values: a) mean in x-direction; b) mean in y-direction; c) mean in z-direction.

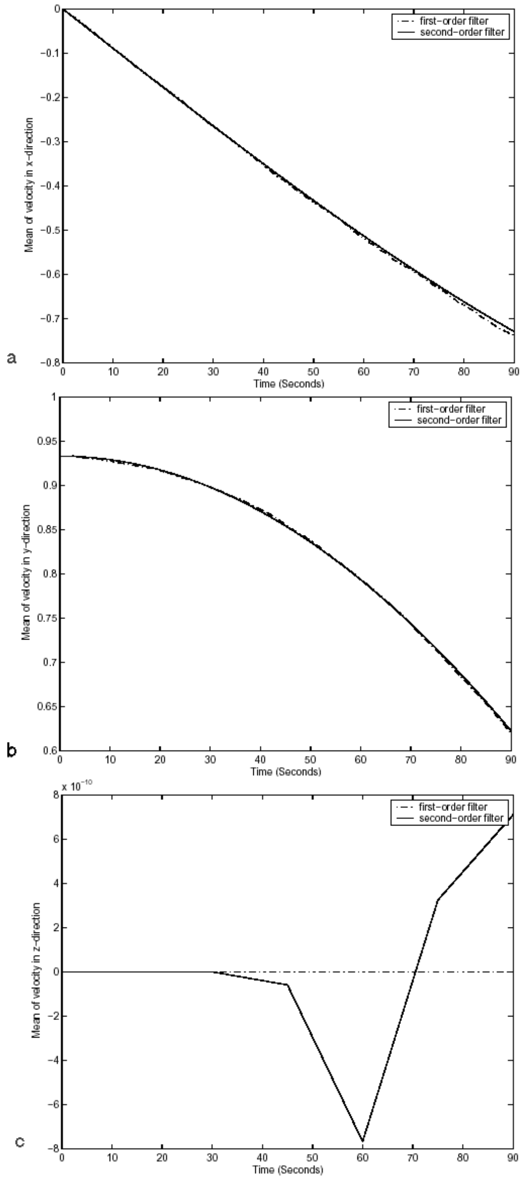

Figure 6: Trajectory II mean values of velocity: a) mean of velocity in x-direction; b) mean of velocity in y-direction; c) mean of velocity in z-direction.

The maximum distance of the x-component of the position of a particle moving along an ellipse is the major axis and the minimum is zero. Likewise, the maximum distance of y-component of the position is minor axis and the minimum is zero. The Figs 3ab and 5ab for both the trajectories show clearly and justify the fact that orbit estimate is close to being elliptical. When the particle is at the end of the major axis, then x-component of velocity is minimum and y-component is maximum. When the satellite moves from the end of the major axis to the end of the minor axis, the x-component of the velocity increases and y-component decreases in magnitude, see Figs 4ab and 6ab. When the satellite finally arrives at the minor axis, x-component of the velocity become maximum and y-component of the velocity becomes minimum. The motion of the satellite in z-direction, i.e. the components $\hat{z}_{t_k}^{t_k}$, and $\hat{v}_{z_{t_k}}^{t_k}$ is attributed to the only driving force caused by the stochastic acceleration and the contribution for the evolution does not come from the initial conditions, since the initial conditions are assumed zero. This illustrate the ability of the second-order nonlinear filter to preserve the perturbation effect acting on the orbiting satellite. Table 1 also indicates that the nonlinear filter of this paper is useful, since it results the SNR larger compared to unity.

5 Conclusion

In this paper, we have developed a second-order nonlinear continuous-discrete filter for the satellite tracking. The main contribution of this paper is to examine the accuracy of the nonlinear filter by considering the 'stochastic differential equation formalism' for the satellite dynamics. Such a dynamics provides an accurate description of reality. The effectiveness of the second-order nonlinear filter is examined on the basis of it's ability to preserve perturbation effect felt by the satellite and to produce larger SNR compared to unity and provides quite simplified analysis. The mean trajectory for the dust perturbed satellite using first-order approximation does not include variance term in mean evolution. The term GQG^T in variance evolution accounts for the stochastic perturbation felt by the orbiting satellite. For this reason, first-order filter does not preserve perturbation effects in mean evolution. In order to account for the stochastic perturbation in mean evolution, we use the second-order filter. The second-order approximation includes the second-order partials of system nonlinearity and variance terms in the mean trajectory, which leads to better estimation of the trajectory.

Appendix A

The Kolmogorov forward equation describes the time evolution of the conditional probability density and subsequently used for obtaining the conditional mean and conditional variance. The Kolmogorov forward equation is given by [1]p. 164

$$dp(\xi_t|\zeta_{t_{k-1}}) = (p(\xi_t|\zeta_{t_{k-1}}))dt \quad t_{k-1} \leq t < t_k$$

where

$$(p(\xi_t|\zeta_{t_{k-1}})) = -\sum_{i=1}^{n} \frac{\partial(pf_i)}{\partial\xi_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2(p(GQG^T)_{ij})}{\partial\xi_i\partial\xi_j}$$

is the diffusion operator.

Let $\varphi(\xi)$ be a twice continuously differentiable scalar function of the n-vector ξ. Defining the expectation of φ using the conditional density $p(\xi_t|\zeta_{tau}), \quad t > \tau$

$$\hat{\varphi}^\tau(\xi_t) = \mathbb{E}^\tau[\varphi(\xi_t)] = [[\varphi(\xi_t)|\zeta_\tau] = \int \varphi(\xi)p(\xi_t|\zeta_\tau)d\xi$$

taking the differential on both sides, we get

$$d\hat{\varphi}^\tau(\xi_t) = \int \varphi(\xi)dp(\xi_t|\zeta_\tau)d\xi$$

$$d\hat{\varphi}^\tau(\xi_t) = \Big(\int \varphi(\xi)(p(\xi_t|\zeta_\tau))d\xi\Big)dt$$

alternatively, the above equation can be stated as

$$d\hat{\varphi}^\tau(\xi_t) = \big(\varphi(p(\xi_t|\zeta_\tau))\big)\hat{}dt$$

Since the Kolmogorov forward operator is adjoint operator, thus we have

$$d\hat{\varphi}^\tau(\xi_t) = \big({}^*\varphi p(\xi_t|\zeta_\tau)\big)\hat{}dt$$

where * is the Kolmogorov backward operator. after simplification, above expression becomes

$$d\hat{\varphi}^t(\xi_t) = \mathbb{E}[\varphi_\xi^T f]dt + \frac{1}{2}tr\mathbb{E}^t[GQG^T\varphi_{\xi\xi}]dt \quad t_{k-1} \leq t < t_k$$

where φ_ξ is the gradient and $\varphi_{\xi\xi}$ is the matrix of second partials. By setting $\varphi(\xi) = \xi_i$ and $\varphi(\xi) = \xi_i\xi_j$, we can determine the evolution of the mean and variance respectively. Thus, we have

$$d\hat{\xi}_t^t/dt = \hat{f}^t(\xi_t, t)$$

$$dP_t^t/dt = \big(\mathbb{E}^t(\xi_t f^T) - \hat{\xi}_t^t \hat{f}^{t^T}\big) + \big(\mathbb{E}^t(f\xi_t^T) - \hat{f}^t \hat{\xi}_t^{t^T}\big) + \mathbb{E}^t(GQG^T) \quad t_{k-1} \leq t < t_k$$

The above two equations are not ordinary differential equations. The right hand side of the above equations involve expectations that require the whole conditional density for their evaluation. Thus, the first two moments of the conditional density depend on all the other moments. Apparently, in order to obtain a computationally realizable and practical filter in the general nonlinear case, some approximation must be made. After introducing second-order approximation, we have

$$d\hat{\xi}_t = [f(\hat{\xi}_t, t) + \frac{1}{2}(P_t \partial^2 f)]dt$$

$$dP_t = [FP_t + P_t F^T + \widehat{GQG^T}]dt$$

where

$$(P\partial^2 f)_i = \sum_{j,k=1}^{n} P_{jk} \frac{\partial^2 f_i(\hat{\xi}, t)}{\partial \xi_j \partial \xi_k}$$

$$F = \left[\frac{\partial f_i(\hat{\xi}, t)}{\partial \xi_j}\right] \quad (n \times n)$$

In component form, the mean and variance evolution equations are

$$d\hat{\xi}_{t_i} = f_i(\hat{\xi}_t, t)dt + \frac{1}{2}\sum_{j,k=1}^{n} P_{jk} \frac{\partial^2 f_i(\hat{\xi}, t)}{\partial \xi_j \partial \xi_k} dt$$

$$(dP_t)_{ij} = \Big(\sum_{k=1}^{n} P_{ik} \frac{\partial f_j(\hat{\xi}, t)}{\partial \xi_k} + \sum_{k=1}^{n} P_{jk} \frac{\partial f_i(\hat{\xi}, t)}{\partial \xi_k} + (\widehat{GQG^T})_{ij}\Big) dt$$

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