Quantum mechanics of electrons in strong magnetic field

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Abstract

A complete description of the quantum mechanics of an electron in magnetic fields is presented. Different gauges are defined and the relations between them are demonstrated.

1 Quasiclassical quantization

The classical Larmor rotation of a charged particle in a homogeneous external magnetic field is presented in Fig. 1

Consider first the quantization in the semiclassical approximation [1, 2]. The motion of a charge in a magnetic field is periodic in the plane perpendicular to the field and, hence, can be quantized by using the standard Bohr-Sommerfeld quantization condition which after the Peierls substitution

$$\mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\mathbf{A}$$

yields

$$\oint \left(m\mathbf{v} - \frac{e}{c}\mathbf{A} \right) \cdot d\mathbf{r} = (n + \gamma)h \tag{1}$$

where the vector potential $\mathbf{A}$ is related to the magnetic field by

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{2}$$

e and c stand for the electron charge and light speed correspondingly. We can take $\gamma = \frac{1}{2}$ as is the case in the original Bohr-Sommerfeld quantization rule, but its real value follows from an exact solution of the Schrödinger equation, which will be discussed later.
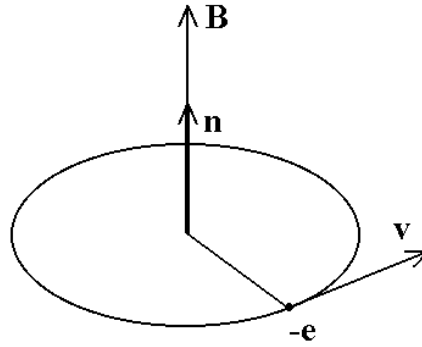

Figure 1: The classical Larmor rotation of a charged particle in a homogeneous external magnetic field.

To estimate the integrals in Eq. (1) we have to take into account that in the plane perpendicular to the magnetic field the classical orbit of a charged particle is a circumference of the Larmor radius ρ_L along which the particle moves with the velocity $\mid \mathbf{v} \mid = \Omega \; \rho_L$, where the Larmor frequency is given by relation $\Omega = \frac{eB}{mc}$. After that the integrals in Eq. (1) can be calculated as follows:

$$\oint \mathbf{A} \cdot d\mathbf{r} = \int [\nabla \times \mathbf{A}] \cdot \hat{n} ds = \pi \rho_L^2 B \; , \tag{3}$$

$$\oint m\mathbf{v} \cdot d\mathbf{r} = 2\pi \rho_L^2 \frac{eB}{c} \; . \tag{4}$$

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Now from Eqs. (1), (3) and (4), it is easy to see that the Larmor radius $\rho_{L,n}$ is quantized, i.e. takes a discrete set of values depending on the integer n :

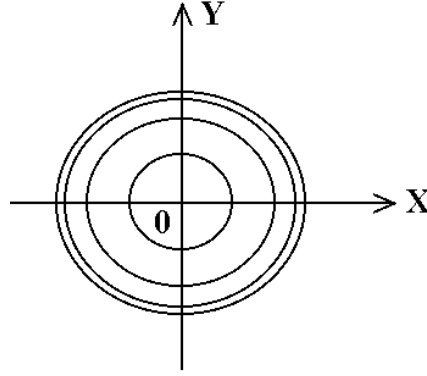$$\rho_{L,n} = \sqrt{\frac{2\hbar}{m\Omega}\left(n+\frac{1}{2}\right)}. \tag{5}$$

Figure 2: Larmor orbits in XY plane.

These quantized electron orbits are shown in Fig. 2. In the quasiclassical limit under consideration (i.e., for $n \gg 1$), the Larmor orbit size depends both on the strength of the magnetic field and the quantum number n. This dependence is given by the relation

$$\rho_{L,n} \propto \sqrt{\frac{n}{B}}\ . \tag{6}$$

Formally, small values of the quantum number n are beyond the scope of the quasiclassical approximation. On the other hand, it is known that in the case of harmonic oscillator the Bohr-Sommerfeld quantization rule gives an exact formula for the energy spectrum. Since the motion of a particle along the circumference with the constant velocity is equivalent to harmonic oscillations, we will see below that one can consider n as an arbitrary integer or zero. Thus, putting $n = 0$ (i.e. in the extreme quantum limit) the Larmor orbit radius becomes equal to

$$\rho_{L,0} = \sqrt{\frac{c\hbar}{eB}} \equiv \sqrt{\frac{\Phi_0}{2\pi B}}. \tag{7}$$

Two fundamental quantities appear in the right hand side of this equation: the magnetic flux quantum $\Phi_0 = hc/e$ which depends only on the world constants and the magnetic length $L_H = \sqrt{\frac{c\hbar}{eB}}$. The flux quantum is the lowest portion by which the magnetic flux through some current carrying loop can be changed. This has a far reaching consequences, as we will see later. But now let us turn to our problem of the energy spectrum calculation. With this end in view, consider the above quantization in the momentum p-space. The classical equation of motion yields

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\left[\mathbf{B} \times \frac{d\mathbf{r}}{dt}\right].\tag{8}$$
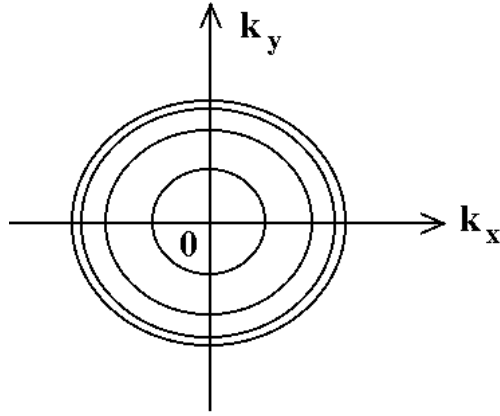

Figure 3: Quantized orbits in $K_x K_y$ plane.

One can see from this equation that: (i) the electron orbit in the p-space is similar to that in the real space (in $x - y$ plane, when field is directed along the $z-$axis as it is shown in Figs. 2 and 3), (ii) in the p-space the orbit is scaled by the factor eB/c, and (iii) the orbit is rotated by $\phi = \pi/2$. Integrating the above equation of motion we obtain

$$p_n = \frac{eB}{c}\rho_{L,n} = \sqrt{2m\hbar\mathbf{\Omega}\left(n + \frac{1}{2}\right)}.\tag{9}$$

We see that quantization of the orbit radius means also quantization of the momentum p which, in turn, implies quantization of the kinetic energy

of the particle. For the quadratic dispersion relation, $E = p^2/2m_e$, we have

$$E_n = \frac{p_n^2}{2m_e} = \hbar\boldsymbol{\Omega}\left(n + \frac{1}{2}\right). \tag{10}$$

This result indicates that the kinetic energy of a charged particle in the $x - y$ plane is quantized in the external magnetic field and the separation between the nearest quantum levels is $\hbar\boldsymbol{\Omega}$. Equation (10) is known in the literature as the Landau formula which describes equidistant quantum spectrum of a charged particle in external magnetic field also called Landau levels.

Thus, the electron motion perpendicular to the magnetic field is quantized, but the motion along the magnetic field is unaffected by the magnetic field and remains free. The corresponding kinetic energy spectrum in the z-direction is given by

$$E_z = \frac{p_z^2}{2m}. \tag{11}$$

Putting together Eqs. (10) and (11) we arrive at the dispersion equation

$$E_n(p_z) = \hbar\boldsymbol{\Omega}\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m}. \tag{12}$$
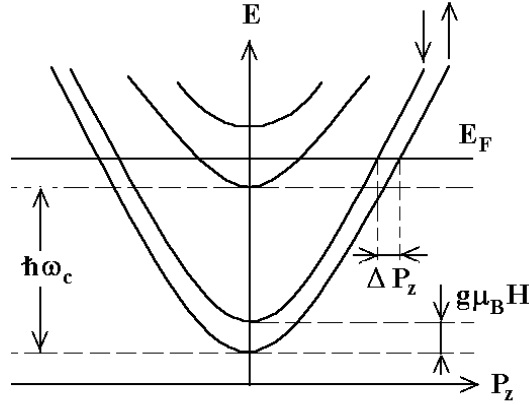
This energy spectrum is shown in Fig. 4.

Figure 4: The Landau levels. Energy dispersion.

To obtain a more concrete idea about the size of the gap between two adjacent Landau levels, we estimate itthe Landau gap. In a typical metal in the field B of the order $10T$ and taking electron mass $m_e \simeq 10^{-27}g$, one can estimate the gap roughly as $\hbar\Omega = \hbar eB/m_e c \simeq 1.5 * 10^{-15} erg \simeq 15K$. In a semiconductor as GaAs, for example, the effective mass of electron may be much smaller , say $m^* \simeq 0.07 m_e$. The Landau gap in this case is of the order of $200K$ in the field of the same strength $B = 10T$. In the GaAs/AlGaAs interface electrons are trapped and behave as a two dimensional electron gas. The energy spectrum of the electrons is completely discrete if the external magnetic field is applied perpendicular to the interface. This results in unusual magnetotransport phenomena which we will discuss later in more detail.

As a result of quantization, all electron states are quenched into the Landau levels. Therefore, each Landau level is highly degenerated. The degeneracy of the Landau level $g(B)$ can be calculated as the number of electronic states between the adjacent Landau levels

$$g\left(B\right) = g_{2d}\hbar\Omega = 2S\frac{B}{\Phi_0}, \tag{13}$$

where

$$g_{2d} = \frac{mS}{\pi\hbar^2} \tag{14}$$

is the density of states for a two dimensional electron gas with quadratic dispersion (see Eq. (12)).

A schematic illustration of the density of states is shown in Fig. 5. We can see from the Eq. (13) that the degeneracy of a Landau level is related to the number of the magnetic flux quantum Φ_0 piercing the sample of area S in a magnetic field B. Factor 2 in the $g\left(B\right)$ comes from the spin degree of freedom. The energy dependence of the 2D density of states shown in Fig. 5 can be obtained by counting the number of states within a thin ring of the width dk and radius $|\mathbf{k}|$ in the k-space.

The density of states in k-space is then $2S/4\pi^2$

$$2\,\frac{S}{4\pi^2}\,2\pi k dk = g(E)dE \ , \tag{15}$$

where the factor 2 accounts for the spin. For free electrons with the quadratic dispersion $E = \hbar^2 k^2/2m$ we obtain the following expression for the density of states

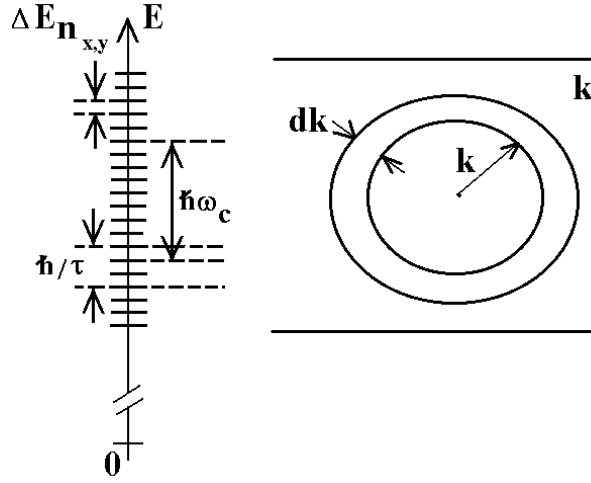$$g(E) = \frac{dN}{dE} = \frac{Sm}{\pi\hbar^2} \ . \tag{16}$$

Figure 5: The density of states for a two dimensional electron gas with quadratic dispersion.

It is clear from Eq. (16), that the density of states for a free electron gas in two dimensions is energy independent. Because the Fermi energy E_F is obtained by filling electron states up from the lowest energy, the E_F is related to the areal density n_s and the density of states as

$$E_F = \frac{n_s}{g_{2d}} = \frac{\pi \hbar^2}{m} n_s. \tag{17}$$

We see, therefore, that the Landau spectrum for free electrons in the 2D case can be obtained on the basis of elementary quasiclassical consideration. In metals and semiconductors the dispersion relation for the quasiparticles, "the conducting electrons", usually far from being a quadratic function of the quasimomentum. The trajectory of the conduction electrons in an external magnetic field is determined by cross section of the Fermi surface, which as a rule is rather complex. Nonetheless, the quasiclassical quantization method works well in this case too and we consider them later in this paper.

1.1 The Landau quantization as a flux quantization problem. The quasiclassical approach. The Lifshitz-Onsager quantization rule

We shall discuss in this section a relation between the Landau quantization and the flux quantization. It is known that the energy of a particle moving along the closed classical orbit becomes quantized in the quantum limit. A free 2D electron with the dispersion relation $E = \mathbf{p}^2/2m$ in external magnetic field moves along the circle of the Larmor radius with the cyclotron frequency $\Omega = eB/mc$. Since the energy is the integral of motion in this case the trajectory of the electron in the momentum space is a circumference of the radius $P = \sqrt{2mE}$ too. The quantization of this motion, as was shown before by many ways, yields the Landau energy spectrum

$$E_n = \hbar\Omega\left(n + 1/2\right). \tag{18}$$

Let us rewrite this formula in the following fashion:

$$S_p(E) = \frac{2\pi\hbar eB}{c}\left(n + 1/2\right) \tag{19}$$

where $S_p(E) = \pi P^2$ is the area of a circle of the radius $P = \sqrt{2mE}$ along which the electron moves in the momentum space. On the other hand, it follows from the classical equation of motion, Eq. (8), that trajectories in the coordinate and momentum spaces are of the same form but turned related to each other by the right angle and scaled by the factor eB/c. The latter means that the radii of the circumferences in the coordinate R and momentum P spaces are related by the condition $R = cP/eB$. Taking this into account we can rewrite the Landau quantization formula in the following fashion:

$$S_R(E) = \frac{2\pi\hbar c}{eB}\left(n + 1/2\right) \tag{20}$$

where $S_R(E) = \pi R^2(E)$ is the area inside the circle of the radius $R(E) = cP(E)/eB$ in the coordinate space. We can calculate then the flux through this circle $\Phi_R(E) = S_R(E)B$ and see that this quantity is quantized

$$\Phi_R(E) = \Phi_0\left(n + 1/2\right) \tag{21}$$

where Φ_0 is the flux quantum. Therefore, we see that both in the coordinate and momentum spaces the Landau quantization means the quantization of the area inside the closed classical trajectory but with the different steps: $\Delta S_R = 2\pi\hbar c/eB$ in the coordinate space and $\Delta S_P = 2\pi\hbar eB/c$ in the momentum space. In the coordinate space the Landau quantization also means

the flux quantization through the closed loop with the quantum Φ_0. The above quantization rules can be easily generalized to the case of an arbitrary electron dispersion which is the usual case in the crystal solids like metals and semiconductors. The quantization of the $S_p(E)$ is known in the literature as the Lifshitz-Onsager quantization rule.

The Lifshitz-Onzager quantization rule is a direct consequence of the commutation rules between the momentum components $\hat{p}_\alpha = (\hbar/i)\partial/\partial x_\alpha + (e/c)A_\alpha$ in the external magnetic field B directed along the Z-axes of the Cartesian coordinate system:

$$[\hat{p}_x, \hat{p}_y] = \frac{e\hbar}{c}B, [\hat{p}_y, \hat{p}_z] = [\hat{p}_x, \hat{p}_z] = 0 \qquad (22)$$

(where A_α is the vector-potential). These equations mean that the momentum $\hat{p}_x$ and the coordinate $\hat{q}_x = c\hat{p}_y/eB$ satisfy the stand commutation rule $[\hat{p}_x, \hat{q}_x] = \hbar/i$ so that the quasiclassical quantization rule holds

$$\oint p_x dq_x = 2\pi\hbar(n + \gamma). \qquad (23)$$

This equation is exactly the Lifshitz-Onzager quantization rule in as much as the integral $\oint p_x dp_y = S_p(E, p_z)$ equals to the cross-section of the Fermi surface by the plane $p_z = $ constant.

$$S(E, p_z) = \frac{2\pi\hbar eB}{c}(n + \gamma). \qquad (24)$$

We can obtain Eq. (24) within the Feynman scheme since only one classical path connects two arbitrary points p_0 and p_1 at the trajectory which is a cross-section of the Fermi surface. The appropriate Feynman amplitude is given by

$$\exp\left(i\frac{S_{cl}}{\hbar}\right) = \exp\left(i\frac{c}{eB\hbar}\int_{p_0}^{p_1} p_x(p_y')dp_y'\right), \qquad (25)$$

where S_{cl} denotes the classical action at the segment between the points p_0 and p_1. If the amplitude to find an electron in the point p_0 at the trajectory is $C(p_0)$ then an amplitude to arrive at the same point of the Fermi-surface cross-section after the one complete rotation is $C(p_0)\exp[i\varphi(E, p_z) + i\alpha]$ where

$$\varphi(E, p_z) = \frac{c}{eB\hbar}\oint p_x(p_y')dp_y' = \frac{c}{eB\hbar}S(E, p_z) \qquad (26)$$

and α is some arbitrary constant. Equating these amplitudes and taking $\alpha = \pi$ we arrive at the Lifshitz-Onsager quantization rule of Eq. (24). In

solids the Fermi surface repeats periodically along the crystal symmetry directions. This means that in the external magnetic field the cross-section of the whole energy surface by the plane p_z =constant yields a network of periodic classical orbits. In some organic conductors and conventional metals this 2D network consists of closed orbits connected by the magnetic breakdown centers. We will consider a generalization of the Lifshitz-Onsager quantization rule to this problem later.

2 Gauge invariant formulation

2.1 The gauge invariance in a classical-analogy approach

In this section we discuss the correspondence between the classical description of rotation of a charged particle in external magnetic field and the elementary quantum mechanical consideration of this motion. We start with the classical description of the Larmor orbit.

Figure 6: The classical Larmor orbit where $\rho_0^2 = x_0^2 + y_0^2$ describes the center of rotation and ρ is the radius vector directed to the rotation point.

In the quantum mechanical approach, the classical dynamic variables are generalized to be the quantum operators. In the classical picture, the Lar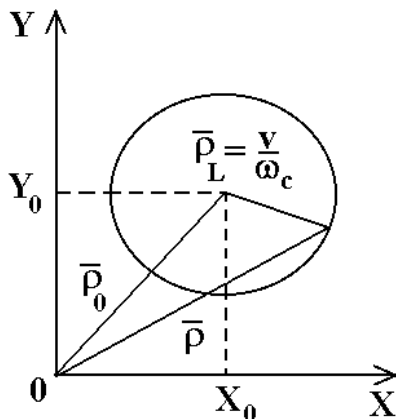mor radius is given by $\rho_L = v/\Omega$. The classical Larmor orbit is shown in Fig. 6, where $\rho_0^2 = x_0^2 + y_0^2$ describes the center of rotation and ρ is the radius vector directed to the rotation point. These vectors are related by

the equation of motion

$$[\Omega \times (\rho - \rho_0)] \;=\; \mathbf{v}_\perp \;. \tag{27}$$

Now turn to the quantum mechanical description in which the classical variables we defined above becomes operators. Proceeding in that fashion we introduce first the operator for the center of orbit

$$\hat{\rho}_0^2 = \hat{x}_0^2 + \hat{y}_0^2 \tag{28}$$

where coordinates and velocity components are Hermitian operators

$$\hat{x}_0 = \hat{x} - \frac{\hat{v}_y}{\Omega} \;, \quad \hat{y}_0 = \hat{y} + \frac{\hat{v}_x}{\Omega} \;. \tag{29}$$

Analogous, the Larmor radius operator is given by

$$\hat{\rho}_L^2 \;=\; \frac{1}{\Omega^2} \left(\hat{v}_x^2 + \hat{v}_y^2 \right) \;. \tag{30}$$

Let us discuss now some necessary for further commutation relations. Using the commutation identity relation

$$[\hat{A}^2, \hat{B}] \equiv \hat{A}[\hat{A}, \hat{B}] + [\hat{A}, \hat{B}]\hat{A} \tag{31}$$

and taking into account that the Hamiltonian of the charged particle in the external magnetic field is given by

$$\hat{H} = \frac{1}{2m} \left(\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}} \right)^2 \tag{32}$$

with the momentum operator $\hat{\mathbf{p}} = \frac{\hbar}{i}\nabla$, we have

$$\hat{\mathbf{v}} = [\hat{H}, \hat{\mathbf{r}}] = \frac{1}{m} \left(\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}} \right). \tag{33}$$

The commutation relation between the components of the velocity operator is given by

$$[\hat{v}_i, \hat{v}_k] = \frac{ie\hbar}{m^2}\epsilon_{ik\ell}B_\ell \;. \tag{34}$$

We have obtained this result by direct calculations:

$$[v_i, v_k] = \frac{ie\hbar}{m^2} \left\{ \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right\} = \frac{ie\hbar}{m^2}\nabla \times A = \frac{ie\hbar}{m^2}\epsilon_{ik\ell}B_\ell \;.$$

It is straightforward to see that the commutation relation for the coordinate operators describing the center of Larmor rotation is given by

$$[\hat{x}_0, \hat{y}_0] = -iL_H^2 \ . \tag{35}$$

The quantity L_H in Eq. (35), the magnetic length, which is determined by $L_H^2 = \hbar c/eB$. The commutation relation between the velocity and coordinate operator components is given by

$$[\hat{v}_i, \hat{x}_k] = -\frac{i\hbar}{m}\delta_{ik} \ . \tag{36}$$

The operators

$$\hat{x}_0 \ , \quad \hat{y}_0 \ , \quad \hat{\rho}_0^2 \ , \quad \hat{\rho}_L^2 \tag{37}$$

are integrals of motion because these operators commute with the Hamiltonian

$$\left[\hat{H}, \hat{x}_0\right] = \left[\hat{H}, \hat{y}_0\right] = \left[\hat{H}, \hat{\rho}_0^2\right] = \left[\hat{H}, \hat{\rho}_L^2\right] = 0, [\hat{\rho}_0, \hat{\rho}_L] = 0 \ . \tag{38}$$

The commutation relations of Eq. (38) are analogous to those of the harmonic oscillator problem. We explore this similarity in what follows for finding the energy spectrum of the electron in a magnetic field. With this purpose in mind we discuss here the correspondence between the Larmor rotation and harmonic oscillator in a more detail.

It is natural to start with the Hamiltonian for the harmonic oscillator problem written in terms of the momentum P and coordinate Q

$$H = \frac{P^2}{2m} + \frac{kQ^2}{2}. \tag{39}$$

In quantum mechanics P and Q become operators $\hat{P}$ and $\hat{Q}$ which satisfy the commutation relation

$$[\hat{P}, \hat{Q}] = -i\hbar \ . \tag{40}$$

The energy spectrum for this problem is then given by

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right), \tag{41}$$

where ω is the oscillator frequency

$$\omega = \sqrt{\frac{k}{m}}. \tag{42}$$

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By using the commutation relations Eq. (38) and Eq. (36), we can make the following correspondence between the Larmor rotation parameters and the operators in the harmonic oscillator problem

$$\hat{v}_x \Leftrightarrow \hat{P} \ ; \quad \hat{v}_y \Leftrightarrow \hat{Q}. \tag{43}$$

The parameters of the Larmor rotation and the harmonic oscillator correspond as follows

$$k \Leftrightarrow \frac{2}{\Omega^2} \ ; \quad \hbar \Leftrightarrow \frac{|e|\hbar B}{m^2 c} \ ; \quad m \Leftrightarrow \frac{\Omega^2}{2}. \tag{44}$$

By using this correspondence relations, it is straightforward to see that we have the energy spectrum $E_n = \hbar\Omega(n + 1/2)$. Because the Hamiltonian for rotating particle is given by

$$\hat{H} = \frac{m\hat{v}_\perp^2}{2} = \frac{m(\hat{v}_x^2 + \hat{v}_y^2)}{2} \ , \tag{45}$$

and taking into account relation $\rho_L = v_\perp/\Omega$, the spectrum of the Larmor radius operator is turned out to be discrete and determined by the equation

$$\left(\rho_L^2\right)_n = \frac{2}{m\Omega^2}E_n = L_H^2(2n + 1) \ . \tag{46}$$

Similarly the discrete spectrum for the center of rotation operator may be obtained by using the definition of the ρ_0^2 and the commutation relation between the coordinates of the center of rotation:

$$(\rho_0^2)_k = L_H^2(2k + 1) \tag{47}$$

The discreteness of the coordinates $(\rho_L)_n$ and $(\rho_0)_n$ yield the following picture for the electron orbits shown in Fig. 7.One can see the manifold of concentric circles of discrete radius $(\rho_0)_n$ centered at the origin of the polar coordinates and with the Cartesian coordinates x_0 and y_0 which define ρ_0. These coordinates are analogous to the q and p operators in the one-dimensional harmonic oscillator problem. Therefore, the Heisenberg uncertainty principle can be applied to them to yield

$$\Delta x_0 \Delta y_0 \geq \frac{L_H^2}{2} \ . \tag{48}$$

The distribution of centers of the electron orbit corresponding to a given ℓ can be represented geometrically as a manifold of circles with the radius
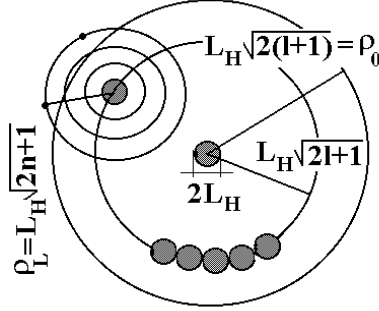
Figure 7: The manifold of concentric circles of discrete radius $(\rho_0)_n$ centered at the origin of the polar coordinates and with the Cartesian coordinates x_0 and y_0 which define ρ_0.

$\simeq L_H^2(2\ell + 1)$ as shown in Fig. 7. The quantum mechanical analog of the Larmor orbit is given by the equation

$$\hat{\rho}_L^2 = (\hat{x} - \hat{x}_o)^2 + (\hat{y} - \hat{y}_o)^2 \tag{49}$$

and the eigenvalues of the operator $\hat{\rho}_L^2$ are determined by the relation

$$(\rho_L^2)_n = (2n + 1)L_H^2, \tag{50}$$

where $n = 0, 1, 2, ...$ The geometric meaning of Eqs. (49) and (50) is illustrated in the Fig.7.

We can introduce now the angular momentum operator related to the Larmor orbital motion as follows

$$L_z = \frac{\hbar L_H^2}{2} \left(\rho_0^2 - \rho_L^2 \right). \tag{51}$$

Its eigenvalues l_z are quantized and given by the relation

$$l_z = sgn(e)\hbar(\ell - n) = \hbar m_z. \tag{52}$$

Summing up the quasiclassical consideration of the Landau problem we must say that quantized Larmor orbitals are only an approximations to the Landau orbitals which can be obtained only on the basis of the Schrödinger equation. This will be done in the next section. But before doing this we consider briefly the gauge invariance in the quantum mechanics.

2.2 The gauge invariance of the Schrödinger equation in an external magnetic field

The phase of the wave function should not affects the observable quantities in quantum mechanics. In particular, the average of the Hamiltonian of a particle,

$$(\Psi, \hat{H}\Psi) = \int \left[\frac{\hbar^2}{2m} \mid \nabla\Psi \mid^2 + U(r) \mid \Psi \mid^2 \right] dr \qquad (53)$$

should be invariant under the substitution $\Psi \rightarrow \Psi e^{i\varphi}$, where $\varphi = \varphi(\mathbf{r})$ is an arbitrary phase. The second term in Eq. (53) is invariant, but the first one is not because of the contribution of the gradient $\nabla\varphi$. To make it invariant we must introduce some vector field $\mathbf{A}$ compensating the gradient term (i.e. containing $\nabla\varphi$) and require that this field does not change the magnetic field $\mathbf{B}$ as a result of the gauge transformations. Both conditions are satisfied under the following substitution:

$$\frac{\hbar^2}{2m} \mid \nabla\Psi \mid^2 \rightarrow \frac{1}{2m} \mid \mathbf{D_A}\Psi \mid^2,$$

where $\mathbf{D_A} = (\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A})$ and the magnetic field is related to the vector $\mathbf{A}$ by the standard equation $\mathbf{B} = rot\mathbf{A}$, so that $\mathbf{B}$ does not changes under the gradient transformation $\mathbf{A} \rightarrow \mathbf{A} + \boldsymbol{\nabla}\chi$ with χ being arbitrary function of $\mathbf{r}$. It is easy to check now that the quantity

$$\int \left[\frac{1}{2m} \mid \mathbf{D_A}\Psi \mid^2 + U(r) \mid \Psi \mid^2 \right] dr$$

is invariant under the gauge transformation $\Psi' = \Psi e^{i\varphi}$ if we also take $\chi = \frac{\hbar}{c}\varphi$ in the gradient transformation $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$. Another words,

$$\frac{1}{2m} \mid \mathbf{D_{A'}}\Psi' \mid^2 = \frac{1}{2m} \mid \mathbf{D_A}\Psi \mid^2 .$$

We see, therefore, that the formal Peierls substitution $\mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\mathbf{A}$, by which a magnetic field is introduced into Hamiltonian dynamics, finds its theoretical justification only on the basis of the quantum mechanics. It is a direct consequence of the gauge invariance under the transformation $\Psi' = \Psi e^{i\varphi}$.

3 The Landau problem in the Landau gauge

Contrary to the elementary consideration of the previous sections which deals primarily with the external magnetic field, the solutions of the Schrödinger

equation depends on the choice of the gauge for the vector potential. Two gauges are most frequently used in the literature: the **Landau gauge** and the symmetric gauge. Let us start with the Landau gauge.

The Schrödinger equation for the charged particle in an external magnetic field B reads

$$\hat{H}\Psi_E = E\Psi_E, \tag{54}$$

where the Hamiltonian of a particle is

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2. \tag{55}$$

In the Landau gauge $\mathbf{A} = (-By, 0, 0)$ only the A_x component of the vector-potential is nonzero, so that $\left[\hat{H}, \hat{p}_x\right] = \left[\hat{H}, \hat{p}_z\right] = 0$. The latter means that the momentum components are the quantum integrals of motion and Ψ_E in Eq. (54) should be also the eigenfunction of the operators $\hat{p}_x$, and $\hat{p}_z$, which means that

$$\Psi_E\left(x, y, z\right) = \varphi_E(y)\exp\left[\frac{i}{\hbar}\left(p_x x + p_z z\right)\right]. \tag{56}$$

Substituting (56) into (54), we have

$$\hat{H}\left(y\right)\varphi_{E'}(y) = E'\varphi_{E'}(y) \tag{57}$$

with

$$\hat{H}\left(y\right) = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\Omega^2}{2}\left(y - y_0\right)^2, \tag{58}$$

$$E' = E - \frac{p_z^2}{2m}, \tag{59}$$

where $\Omega = eB/mc$ is the cyclotron frequency, and

$$y_0 = -cp_x/Be \tag{60}$$

denote the coordinate of the center of the Landau orbit.

Equations (57) and (58) shows the equivalency of the Landau problem to the problem of the quantum oscillator.

Let us introduce a dimensionless variable

$$q = \sqrt{\frac{m\Omega}{\hbar}}\left(y - y_0\right) = \frac{y - y_0}{L_H}. \tag{61}$$

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The quantity $L_H = \sqrt{\hbar c/eH}$ known as the magnetic length, plays an important role of a spatial scale in different problems. We found that the Schrödinger equation for the charged particle in an external magnetic field with the Hamiltonian of Eq. (55) can be written as follows

$$\hat{H}\Psi_{E_n,p_x,p_z} = E_n(p_z)\Psi_{E_n,p_x,p_z}. \tag{62}$$

The eigenvalues of the Eq. (81) are known in the literature as the Landau energy spectrum

$$E_n(p_z) = \hbar\Omega(n + \frac{1}{2}) + \frac{p_z^2}{2m} \tag{63}$$

and the corresponding eigenfunctions are given by

$$\Psi_{E_n,p_x,p_z} = \varphi_n\left(\frac{y - y_0}{L_H}\right)\exp\frac{i}{\hbar}\left(p_x x + p_z z\right). \tag{64}$$

These wave functions depend on a three quantum numbers n, p_x and p_z whereas the energy $E_n(p_z)$ only on the two of them. This means the degeneracy of the spectrum on the momentum p_x which physically is due to the independence of the Landau levels $E_n(p_z)$ on the position of the Larmor orbit center $y_0 = -\frac{cp_x}{eB}$.

The degeneracy $g(B)$ of the Landau level $E_n(p_z)$ can be calculated as a number of states belonging to $E_n(p_z)$ and having different values of the momentum p_x, which yields:

$$g(H) = \frac{L_x\Delta p_x}{2\pi\hbar} = \frac{L_x L_y Be}{2\pi\hbar c} = \frac{\Phi}{\Phi_0}. \tag{65}$$

Here L_x, L_y are the dimensions of a sample in the plane perpendicular to the field **B**, $\Delta p_x = L_y Be/c$ is the maximal value which the component p_x can take (it corresponds to the extreme limit for the Larmor orbit position $y_0 = L_y$), $\Phi_0 = \hbar c/e$ is the flux quantum, and $\Phi = L_x L_y B$ is the total flux through the sample.

The wave function $\varphi_n(q)$ oscillates due to the oscillations of the Hermitian polynomials $H_n(q)$ which have n zeros as a function of the variable q. This is a manifestation of the so called oscillation theorem. This theorem says that the number of zeroth of the wave function is equal to the number of the energy level n of a particle in the potential well counting from the ground state and provided that $n = 0$ is prescribed to the ground state. One can easily calculate a few first polynomials $H_n(q)$ directly from the definition of Eq. (80) to obtain:

$$H_0(q) = 1, \; H_1(q) = 2q, \; H_2(q) = 4q^2 - 2.$$

3.1 The density of states

Consider now another important characteristic of the Landau problem - the density of states. According to the definition the density of states can be calculated as a sum over the Landau energy spectrum

$$g(E) = 2\frac{\Phi}{\Phi_0} \sum_n \int \frac{L_z dp_z}{2\pi\hbar} \delta \left(E - E_n(p_z) \right). \tag{66}$$

The factor $2\frac{\Phi}{\Phi_0}$ appears here because of the degeneracy of the Landau levels on the spin and orbit position. The integration on p_z in Eq. (66) is trivial because of the delta-function. Completing it, we have

$$g(E) = \sum_n g_n(E). \tag{67}$$

The quantity $g_n(E)$ is the density of states in three dimensions for the Landau level with the quantum number n

$$g_n(E) = \frac{V\sqrt{2}m^{3/2}}{\pi^2\hbar^3} \frac{\hbar\Omega}{\left| \sqrt{E - \hbar\Omega \left(n + \frac{1}{2} \right)} \right|}, \tag{68}$$

where V is the volume of the sample. We see that the density of states $g\left(E\right)$ in the Landau problem has a periodic set of the square-root singularities. This type of the singularity is typical for a one-dimensional system. Thus, for fixed value of the quantum number n a motion of electron is effectively one-dimensional. We, therefore, may calculate that external magnetic field effectively reduces the dimensionality of the system.

3.2 The momentum representation

It is well known that the unitary transformation does not change the eigenvalues of the Hamiltonian. On the other hand, sometimes a proper choice of the unitary transformation makes a solution of the eigenproblem much more easier. Many eigenvalue problems become simply in the momentum representation since this approach is, in essence, nothing but a Fourier method known in conventional mathematical physics. In this connection it is interesting to note that in the Landau problem the Hamiltonian given by Eq. (58) is invariant with respect to the momentum representation. Indeed, in

the momentum representation the momentum operator becomes just a variable, $\hat{p} = p$, while the coordinate becomes a differential operator $\hat{y} = i\hbar\partial/\partial p$. Thus, making first the coordinate shift $y + y_0 \to y$ we can rewrite the Hamiltonian of Eq. (58) in the momentum representation as follows

$$\hat{H} = \frac{p^2}{2m} - \frac{m\Omega^2\hbar^2}{2}\frac{\partial^2}{\partial p^2}. \tag{69}$$

This Hamiltonian, after the substitution

$$q = p/\sqrt{m\hbar\Omega}, \tag{70}$$

and introduction of the operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right), \ \hat{a}^+ = \frac{1}{\sqrt{2}}\left(q - \frac{d}{dq}\right), \ \left[\hat{a}, \hat{a}^+\right] = 1$$

takes exactly the form of Eq. (72). Therefore, there is no need to solve the problem anew since the energy spectrum and the wave functions remain the same with the only difference that q in $\varphi_n(q)$ of Eq. (81) is given now by Eq. (70).

For further consideration it is useful to define a couple of Hermitian conjugate operators $\hat{a}$ and $\hat{a}^+$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right), \ \hat{a}^+ = \frac{1}{\sqrt{2}}\left(q - \frac{d}{dq}\right), \ \left[\hat{a}, \hat{a}^+\right] = 1. \tag{71}$$

In terms of these operators the Hamiltonian (58) takes a very simple quadratic form

$$\hat{H} = \hbar\Omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right). \tag{72}$$

We begin the analysis of this Hamiltonian from the definition of the lowest energy eigenstate state, also known in the quantum theory as the ground state. To find the ground state let us consider the average of the Hamiltonian (58)

$$\left(\varphi_E, \hat{H}\varphi_E\right) = \frac{\hbar\Omega}{2}\left(\varphi_E, \varphi_E\right) + \hbar\Omega\left(\hat{a}\varphi_E, \hat{a}\varphi_E\right), \tag{73}$$

which should be real and positive number since $\hat{H}$ is Hermitian operator. The ground state wave function must minimize Eq. (73) and one can easily conclude that this holds under the condition

$$\hat{a}\varphi_0 = 0, \tag{74}$$

which nullifies the second term in the right-hand-side of the Eq. (73) and thereby minimizes quantity $\left(\varphi_E, \hat{H}\varphi_E\right)$. The differential equation (74) has a trivial solution, which, after normalization by the condition $(\varphi_0, \varphi_0) = 1$, yields

$$\varphi_0(q) = \pi^{-1/4} \exp\left(-\frac{q^2}{2}\right). \tag{75}$$

One can check then by a direct substitution, that the wave function

$$\varphi_n = \frac{\hat{a}^{+^n}}{\sqrt{n!}}\varphi_0 \tag{76}$$

is the normalized eigenfunction of the Hamiltonian (72)

$$\hat{H}\varphi_n = E_n\varphi_n \tag{77}$$

with the eigenvalue

$$E_n = \hbar\Omega\left(n + \frac{1}{2}\right). \tag{78}$$

The explicit form for the function $\varphi_n(q)$ directly follows from Eqs. (71) and (76) which yield

$$\varphi_n(q) = \frac{1}{\sqrt{n!2^n\pi^{1/2}}}\left(q - \frac{d}{dq}\right)^n e^{-\frac{q^2}{2}}. \tag{79}$$

One can rewrite the wave function of Eq. (79) in a conventional form with the help of the Hermitian polynomials $H_n(q)$

$$H_n(q) = (-1)^n \left(e^{q^2}\frac{d^n}{dq^n}e^{-q^2}\right). \tag{80}$$

The final result is:

$$\varphi_n(q) = \frac{H_n(q)}{\sqrt{n!2^n\pi^{1/2}}}e^{-\frac{q^2}{2}}. \tag{81}$$

Let us summing up the results for the Landau problem.

3.3 The uncertainty principle in the Landau problem

In this section we shall discuss briefly the uncertainty principle for the coordinate and momentum in the Landau problem. Consider for simplicity the ground state. The wave function of the ground state both in the coordinate and momentum representation is

$$\varphi_0(q) = \pi^{-1/4} \exp\left(-\frac{q^2}{2}\right) \tag{82}$$

with $q = \frac{y - y_0}{L_H}$ in the coordinate representation and $q = \frac{pL_H}{\hbar}$ in the momentum representation. This means that the coordinate uncertainty (the width of a strap in the y-axes direction where the probability to find a particle is appreciable) equals approximately to the $\Delta y \simeq L_H$. The corresponding uncertainty in the momentum of a charged particle in the external magnetic field is $\Delta p \simeq \hbar/L_H$. Thus, the product of these uncertainties equals to

$$\Delta p \Delta y \simeq \hbar. \tag{83}$$

Having at hand the wave functions we can calculate the above uncertainties exactly. To do this we will proceed in such a fashion. Firstly, it follows directly from the definition of Eq. (76) that the eigenfunctions φ_n obey a simple recurrent relations:

$$\hat{a}^+ \varphi_n = \sqrt{n+1}\varphi_{n+1}, \tag{84}$$

$$\hat{a}\varphi_n = \sqrt{n}\varphi_{n-1}. \tag{85}$$

On the other hand, from Eq. (71) we have a couple of equations which express the coordinate and the momentum operators in terms of the quantities $\hat{a}^+$ and $\hat{a}$:

$$\hat{y} = \frac{L_H}{\sqrt{2}}(\hat{a}^+ + \hat{a}), \tag{86}$$

$$\hat{p}_y = \frac{\hbar}{iL_H\sqrt{2}}(\hat{a}^+ - \hat{a}). \tag{87}$$

Note that Eqs. (86)-(87) are valid both in the coordinate and momentum representations. Using these equations and taking into account the orthogonality of the basis φ_n, we can calculate the uncertainties in question with the help of the formal quantum mechanical definitions:

$$\Delta y_n = \sqrt{(\varphi_n, \hat{y}^2\varphi_n) - (\varphi_n, \hat{y}\varphi_n)^2}, \tag{88}$$

$$\Delta p_n = \sqrt{(\varphi_n, \hat{p}^2 \varphi_n) - (\varphi_n, \hat{p} \varphi_n)^2}. \tag{89}$$

Elementary calculations then yields

$$\Delta y_n = L_H \sqrt{n + \frac{1}{2}}, \tag{90}$$

$$\Delta p_n = \hbar / L_H \sqrt{n + \frac{1}{2}}. \tag{91}$$

Thus, the uncertainty principle for the arbitrary n state in our problem reads

$$\Delta y_n \Delta p_n = \hbar \left(n + \frac{1}{2} \right). \tag{92}$$

Putting the integer $n = 0$ in Eq. (92) we see that our qualitative estimation of uncertainties for the ground state (83) only by the factor one half differs from the exact formula (92).

4   The coherent state

The eigenfunction of the operator $\hat{a}$ defined in the previous section is known in the literature as the coherent state which minimize the product of uncertainties of the coordinate and momentum of a particle. Let us define the coherent state ψ_α by the equation

$$\hat{a} \psi_\alpha = \alpha \psi_\alpha. \tag{93}$$

The eigenvalue α is a complex number since $\hat{a}$ is non-Hermitian operator. Writing ψ_α in the Landau basis φ_n, we have

$$\psi_\alpha = \sum_n C_\alpha(n) \varphi_n. \tag{94}$$

Taking advantage of relations (84) and (85) we arrive at the following recurrent equation for the coefficient $C_\alpha(n) = (\varphi_n, \psi_\alpha)$:

$$C_\alpha(n) = \frac{\alpha}{\sqrt{n}} C_\alpha(n-1). \tag{95}$$

Using the recurrent equation (95) and normalizing the coherent state by condition $(\psi_\alpha, \psi_\alpha) = 1$ we may write ψ_α as a series

$$\psi_\alpha = \exp\left(-\frac{|\alpha^2|}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \varphi_n, \qquad (96)$$

where $\exp\left(-\frac{|\alpha|^2}{2}\right)$ is the normalization coefficient.

One can recast the coherent state vector (96) in a more compact form

$$\psi_\alpha = e^{\alpha \hat{a}^+ - \alpha^* \hat{a}} \varphi_0. \qquad (97)$$

by taking advantage the well-known operator identity

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \qquad (98)$$

which holds because $\hat{A} = \alpha \hat{a}^+$ and $\hat{B} = -\alpha^* \hat{a}$ in our case and the commutator $\left[\hat{A}, \hat{B}\right] = |\alpha|^2$ is the c-number (not operator).

It is easy to check straightforward that the coherent states are nonorthogonal

$$(\psi_\alpha, \psi_\beta) = \exp\left(-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + \alpha^*\beta\right)\right). \qquad (99)$$

In the case of $|\alpha - \beta| \gg 1$ they are approximately ortogonal in as much as the absolute value of the scalar product (99) is small

$$|(\psi_\alpha, \psi_\beta)| = \exp\left(-\frac{1}{2}|\alpha - \beta|\right) \ll 1. \qquad (100)$$

The set of the coherent states is complete. The completeness means that following identity holds for the wave vector $\psi_\alpha(q)$:

$$\frac{1}{\pi} \int d^2\alpha \psi_\alpha^*(q) \psi_\alpha(q') = \delta\left(q - q'\right). \qquad (101)$$

This expression immediately comes out from the Eq. (96) and the completeness of the Landau basis functions $\varphi_n(q)$:

$$\sum_n \varphi_n(q)\varphi_n(q') = \delta\left(q - q'\right). \qquad (102)$$

All the equations considered so far are valid both in the coordinate and momentum presentations. In the coordinate presentation the quantity q is given by the Eq. (61) and in the momentum presentation by Eq. (70) .

Using an explicit form for the Landau basis functions $\varphi_n(q)$ (see Eq. (81)) and substituting them into the Eq. (96), we have

$$\psi_\alpha(q) = \pi^{-1/4} \exp\left(-\frac{|\alpha|^2}{2} - \frac{q^2}{2}\right) \sum_{n=0}^{\infty} \left(\frac{\alpha}{\sqrt{2}}\right)^n \frac{H_n(q)}{n!}. \qquad (103)$$

The sum in Eq. (103) can be easily calculated with the help of the generic function relation for the Hermitian polynomials

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \qquad (104)$$

This equation simply means that the coefficients of the power series expansion with respect to the variable t for the function standing in the left-hand-side are equal to the Hermitian polynomials given by Eq. (80). This statement can be easily checked by direct calculations. Thus, taking into account the Eq. (104) one may recast the Eq. (103) into the following Gauss-like form

$$\psi_\alpha(q) = \pi^{-1/4} \exp\left(-\frac{|\alpha|^2}{2} + \frac{\alpha^2}{2}\right) \exp\left(-\left(\frac{q}{\sqrt{2}} - \alpha\right)^2\right). \qquad (105)$$

This Gauss-like wave function is known to minimize the uncertainty relation for the coordinate and momentum (i.e. makes the right hand side in the equation $\Delta y_\alpha \Delta p_\alpha = \hbar/2$ exactly equal to the lowest value $\hbar/2$).

Such a wave function was first introduced by Schrödinger under the name of a coherent state. We see that the Schrödinger definition of the coherent state and that given by the Eq. (93) are identical in essence.

One of the practical advantages of the coherent state ψ_α is that the matrix elements of the type $\left(\psi_\beta, \hat{a}^{+n}\hat{a}^m\psi_\alpha\right)$ can be calculated very easy in the basis of the coherent states. For example, it is easy to check that

$$\left(\psi_\beta, \hat{a}^{+n}\hat{a}^m\psi_\alpha\right) = \left(\hat{a}^n\psi_\beta, \hat{a}^m\psi_\alpha\right) = (\beta^*)^2 \alpha^m \left(\psi_\beta, \psi_\alpha\right). \qquad (106)$$

With the help of this equation we have from Eq. (72)

$$\left(\psi_\alpha, \hat{H}\psi_\alpha\right) = \hbar\Omega\left(|\alpha|^2 + \frac{1}{2}\right). \qquad (107)$$

On the other hand, the average values of the operators $\hat{x}$ and $\hat{p}$ in the coherent state ψ_α are equal to

$$(\psi_\alpha, \hat{x}\psi_\alpha) = \left(\frac{2\hbar}{m\Omega}\right)^{1/2} Re\alpha; \quad (\psi_\alpha, \hat{p}\psi_\alpha) = (2m\hbar\Omega)^{1/2} Im\alpha. \qquad (108)$$

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Combining Eq. (107) and (108) we obtain

$$\left(\psi_\alpha, \hat{H}\psi_\alpha\right) = \frac{m\Omega^2}{2}\left(\psi_\alpha, \hat{x}\psi_\alpha\right)^2 + \frac{1}{2m}\left(\psi_\alpha, \hat{p}\psi_\alpha\right)^2 + \frac{\hbar\Omega}{2}. \qquad (109)$$

This expression, written in terms of averaged $\hat{x}$ and $\hat{p}$ operators, is very similar to the energy of the classical oscillator

$$E = \frac{m\Omega^2}{2}x^2 + \frac{p^2}{2m}$$

and demonstrates a closeness of the coherent state description to the classical approach. Another manifestation of this, as was note above, is the fact that ψ_α minimizes the uncertainty principle for the coordinate and the momentum. To see this, we can directly calculate these uncertainties in the coherent state, which yields:

$$\delta x_\alpha = \sqrt{(\psi_\alpha, \hat{x}^2\psi_\alpha) - (\psi_\alpha, \hat{x}\psi_\alpha)^2} = \left(\frac{\hbar}{2m\Omega}\right)^{1/2}, \qquad (110)$$

$$\delta p_\alpha = \sqrt{(\psi_\alpha, \hat{p}^2\psi_\alpha) - (\psi_\alpha, \hat{p}\psi_\alpha)^2} = \left(\frac{m\hbar\Omega}{2}\right)^{1/2}. \qquad (111)$$

Multiplying these quantities, we have

$$\delta x_\alpha \delta p_\alpha = \frac{\hbar}{2}. \qquad (112)$$

Comparing this result with uncertainty relation in the Landau basis given by Eq. (92) we see that only the ground state $n = 0$ minimizes the uncertainties product for the coordinate and momentum, since the ground state $\varphi_0(q)$ is exactly Gaussian in shape, i.e. it is the coherent wave function according to the Schrödinger definition.

In as much as the coherent state ψ_α was presented above as a series of the Landau states (see Eq. (96)) it is easy to write down the probability distribution for the Landau quantum number n in the coherent state:

$$W_\alpha(n) = |C_\alpha(n)|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \qquad (113)$$

Taking into account that $|\alpha|^2 = \overline{n}_\alpha$, where $\overline{n}_\alpha = (\psi_\alpha, \hat{a}^+\hat{a}\psi_\alpha)$ stands for the average of the quantity $\sqrt{\sqrt{\sqrt{}}}\, n$ in the ψ_α-state, we see that Eq. (113) is nothing but the Poisson probability distribution function

$$W_\alpha(n) = e^{-\overline{n}_\alpha} \frac{\overline{n}_\alpha^n}{n!}. \tag{114}$$

5 The symmetric gauge in the Landau problem

Because the external magnetic field imposes an axial symmetry to the Landau problem, it is natural to solve the Schrödinger equation in the cylindrical coordinates. The vector potential in the symmetric gauge is defined as follows $\mathbf{A} = \frac{1}{2}[\mathbf{Br}]$. The symmetric gauge is very popular, for example, in the theory of interacting many-body systems in a magnetic field. Thus, we shall use the cylindrical coordinates (ρ,ϕ,z), in the Schrödinger equation for a charged particle of the mass m_e in a magnetic field described by the symmetric gauge

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\Psi - \frac{i\hbar\Omega}{2}\frac{\partial\Psi}{\partial\phi} + \frac{m^*\Omega^2}{8}\rho^2\Psi = E\Psi. \tag{115}$$

The magnetic field here is assumed to be parallel to the z-axis and we omit for brevity terms relating to the free motion along the field direction. The above Schrödinger equation can also be obtained in the cylindrical coordinates if we choose the vector potential as

$$A_\phi = \frac{1}{2}\rho B , \quad A_\rho = A_z = 0 , \tag{116}$$

which is just another form of the symmetric gauge.

Because the coefficients in the differential Eq. (115) depend only on the radial coordinate ρ, the momentum $\hat{p}_z = \hbar\hat{k}_z$ and angular momentum $\hat{l}_z = i\hbar\partial/\partial\varphi$ are the quantum integrals of motion so that the solution can be factorized to separate the variables

$$\Psi = \Psi_\phi\Psi_z R(\rho). \tag{117}$$

Here Ψ_z is the plane wave along the z-axis (the eigenfunction of the $\hat{p}_z = \hbar\hat{k}_z$)

$$\Psi_z = e^{ik_z z}, \tag{118}$$

and Ψ_ϕ is the eigenfunction of the operator $\hat{l}_z$

$$\Psi_\phi = e^{im\phi} \tag{119}$$

with the eigenvalue $\hbar\,m$ where m is an integer. The radial part of the wave function $R(\rho)$ satisfies the equation

$$\frac{\hbar^2}{2m_e}\left[\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho}\right] + f(\rho)R = 0 \qquad (120)$$

and the function $f(\rho)$ is given by

$$f(\rho) \equiv -\frac{\hbar^2}{2m_e}\frac{m^2}{\rho^2} + E - \frac{p_z^2}{2m_e} - \frac{1}{8}m_e\Omega^2\rho^2 - \frac{\hbar\Omega}{2}m. \qquad (121)$$

We rewrite now the Eq. (120) in a dimensionless form

$$\tilde{\rho}^2 R'' + R' + \left(-\frac{1}{4}\tilde{\rho}^2 + \eta - \frac{m^2}{4\tilde{\rho}^2}\right)R = 0, \qquad (122)$$

where a parameter η does not depend on the dimensionless coordinate $\tilde{\rho} = \rho/L_H$

$$\eta = \frac{1}{\hbar\Omega}\left(E - \frac{p_z^2}{2m^*}\right) - \frac{m}{2} \qquad (123)$$

and derivatives are taken with respect to $\tilde{\rho}$. (L_H stands for the magnetic length).

To determine the radial part $R(\rho)$ of the wave function it is instructive first consider the limiting cases of large and small $\tilde{\rho}$. We see that if $\tilde{\rho} \to \infty$, the wave function exponentially decreases as $\Psi \propto e^{-\tilde{\rho}^2/2}$, while near the origin (i.e., when $\tilde{\rho} \to 0$), the asymptotic behavior becomes power-like $\Psi \propto \tilde{\rho}^{|m|}$. This prompts us to write a solution for the $R(\tilde{\rho})$ in the form

$$R(\tilde{\rho}) = e^{-\frac{\tilde{\rho}^2}{2}}\tilde{\rho}^{|m|}u(\tilde{\rho}) . \qquad (124)$$

By substituting Eq. (124) into Eq. (122), we find that $u(\tilde{\rho})$ can be expressed in terms of the degenerate hypergeometric function $F(\alpha, \eta, z)$ since it satisfies the following differential equation

$$zu'' + (\eta - z)u' - \alpha u = 0. \qquad (125)$$

The degenerate hypergeometric function is determined by the series in variable z

$$F(\alpha, \eta, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\eta)_k}\frac{z^k}{k!},$$

where $(\alpha)_k$ and $(\gamma)_k$ stand for the product of the form $(\alpha)_k = \alpha(\alpha+1)...(\alpha+k)$. It has the following properties: (a) the series in z converges only for a

finite value of z, (b) η should not take neither zero nor negative integer values, and (c) α is an arbitrary value, (d) $F(\alpha, \eta, z)$ is polynomial when α is a negative integer. In our case the function $u(\tilde{\rho})$ satisfies Eq. (125) so that its solution is given by

$$u = F\left(\alpha, \eta, z\right) \tag{126}$$

with

$$\alpha \equiv \left(\frac{|m| + 1 - m}{2} + \frac{p_z^2/2m_e - E}{\hbar\Omega}\right), \tag{127}$$

$$\eta \equiv |m| + 1, \tag{128}$$

and $z = \tilde{\rho}^2$. Combining these results and normalizing R by the condition $\int_0^\infty R^2 \rho d\rho = 1$, we obtain the radial wave function in the form

$$R_{n_\rho, m}(\rho) = \frac{1}{L_H^{|m|+1} m!} \sqrt{\frac{(|m| + n_\rho)!}{2^{|m|} n_\rho!}} e^{-\frac{\rho^2}{4L_H^2}} \rho^{|m|} F\left(-n_\rho, |m| + 1, \frac{\rho^2}{2L_H^2}\right). \tag{129}$$

The energy spectrum is determined by the condition of the finiteness of the wave function, which holds if α is a nonzero negative integer, say n_ρ. This condition defines the energy levels as follows

$$E = \hbar\Omega\left(n_\rho + \frac{|m| + m + 1}{2}\right) + \frac{\hbar^2 k_z^2}{2m_e}. \tag{130}$$

It is useful to introduce a new quantum number

$$n = n_\rho + \frac{|m| + m}{2}.$$

Then the energy spectrum acquire the standard form of the Landau spectrum

$$E_n(k_z) = \hbar\Omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m_e}.$$

Each level has an infinite degeneracy since for fixed integer n the orbital number m takes values from $-\infty$ to n. Putting $n_\rho = 0$, and assuming m to be positive (which means that $n = m$) we note that the radial component of the wave function can be rewritten as a function of the complex coordinate $z = x + iy$ in the following form

$$\Psi_n(z) = \left(\frac{1}{\pi L_H^2 2^{n+1} n!}\right)^{\frac{1}{2}} \left(\frac{z}{L_H}\right)^n e^{-\frac{|z|^2}{4L_H^2}}. \tag{131}$$

In the quasiclassical limit (i.e., for large $n \gg 1$), the electron wave function localized mainly within a ring of the width L_H and radius $L_H \sqrt{2n}$. One can see this after writing down the radial coordinate probability distribution function

$$|\Psi_n|^2 = C^2 \left(\frac{\rho}{L_H}\right)^{2n} e^{-\frac{\rho^2}{2L_H^2}} \ . \tag{132}$$

Calculating then the expectation values of the radius and its square, with the help of this function, we have

$$\langle \rho \rangle = 2\pi \int_0^\infty \rho \, |\Psi_n|^2 \, \rho d\rho = L_H \sqrt{2} \, \frac{\Gamma\left(n + \frac{3}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right)} \ , \tag{133}$$

$$\langle \rho^2 \rangle = 2\pi \int_0^\infty \rho^2 \, |\Psi_n|^2 \, \rho d\rho = 2L_H^2(n+1) \ . \tag{134}$$

The normalization constant C is given by the equation $C^2 = L_H^2 \pi / 2^{n+1} \Gamma(n+1)$. For large $n \gg 1$, we obtain

$$\langle \rho \rangle \simeq L_H \sqrt{2n} \ , \tag{135}$$

$$\sqrt{\langle \rho^2 \rangle} \simeq \langle \rho \rangle \gg L_H \ . \tag{136}$$

On the other hand, the radial coordinate probability distribution function $2\pi\rho \, |\Psi_n|^2$ has a narrow peak of the width L_H centered at $\rho_0^2 = L_H^2(2n+1)$. To see this we can do following elementary transformations:

$$2\pi\rho \, |\Psi_n|^2 = C \left(\frac{\rho}{L_H}\right)^{2n+1} \exp(-\frac{\rho^2}{2L_H^2}) = C \exp\left[-G(\frac{\rho}{L_H})\right], \tag{137}$$

where

$$G(\frac{\rho}{L_H}) = \frac{\rho^2}{2L_H^2} - (2n+1)\ln(\frac{\rho}{L_H}).$$

This function has a minimum at $\rho_L^2 = L_H^2(2n+1)$. Expanding then $G(\frac{\rho}{L_H})$ in the power series near the ρ_0, we have

$$2\pi\rho \, |\Psi_n|^2 \propto \exp[-\frac{(\rho - \rho_L)^2}{L_H^2}].$$

The above relations mean that in quasiclassical limit a charged particle moves most probably within the ring strap of the radius $L_H \sqrt{2n}$ and the width $L_H << L_H \sqrt{2n}$ as it is shown in the Fig. 8.

The energy levels of a charged particle in a uniform magnetic field can be obtained by using the semi-classical approximation. According to the
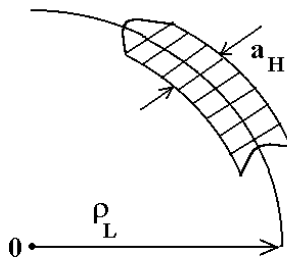
Figure 8: In a quasiclassical limit a charged particle moves most probably within the ring strap of the radius $L_H \sqrt{2n}$ and the width $L_H \ll L_H \sqrt{2n}$.

Eq. (120) we can write down $f(\rho) = E - \frac{p_z^2}{2m_e} - V_{eff}(\rho)$, where the effective potential for the Schrödinger equation in our problem takes the form

$$V_{eff}(\rho) = \frac{\hbar^2}{2m_e\rho^2}\left(m + \frac{eB}{2\hbar c}\rho^2\right)^2. \tag{138}$$

The elementary analysis then shows that the classically accessible region of the radial motion of a particle in a magnetic field is given by

$$\rho_{1,2} = \sqrt{\frac{2\hbar c}{eB}}\left[\sqrt{n + m + \frac{1}{2}} \pm \sqrt{n + \frac{1}{2}}\right]. \tag{139}$$

In the extreme quantum limit (i.e., $n = 0$), the wave function is given by

$$|\Psi_{n=0,m}|^2 = C^2\left(\frac{\rho}{L_H}\right)^{2|m|} e^{-\frac{\rho^2}{2L_H^2}}. \tag{140}$$

Since the shape of this wave function coincides with that of Eq. (132) we conclude that the particle motion for $m \gg 1$ looks like a ring of $L_H\sqrt{m}$ and width L_H.

The electrical current related to the Landau orbitals can be easily calculated on the basis of the standard equation

$$\mathbf{j} = \frac{ie\hbar}{2m_e}(\Psi\boldsymbol{\nabla}\Psi^* - \Psi^*\boldsymbol{\nabla}\Psi) - \frac{e^2}{m_e c}\mathbf{A}\Psi\Psi^*. \tag{141}$$

It is clear that radial component of the current is equal zero in view of the particle localization within the plane perpendicular to the magnetic

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field $\mathbf{B}$, i.e. $\mathbf{j}_\rho = 0$. Computing the current in the other two directions of the cylinder coordinate system in the symmetric gauge, we have

$$\mathbf{j}_\varphi = \left(\frac{e\hbar m}{m_e \rho} - \frac{e^2 B}{2 m_e c} \rho \right) |\Psi_{nmk_z}|^2 \ , \tag{142}$$

$$\mathbf{j}_z = \frac{e p_z}{m_e} |\Psi_{nmk_z}|^2 . \tag{143}$$

In the following subsections, we discuss another examples where the symmetric gauge is convenient to apply for solving the Schrödinger equation.

6 Coherent state in the symmetric gauge

In this section we shall generalize the coherent state introduced previously in the Landau gauge to the case of the symmetric gauge. To do this it is necessary again to obtain a solution of the Schrödinger equation in the symmetric gauge, but this time in the operator form which generalizes the approach of the section 2.5.

The Hamiltonian of a charged particle in the symmetric gauge

$$\mathbf{A} = \frac{1}{2} \left(-By, Bx, 0 \right) \tag{144}$$

takes the form

$$\hat{H}_t = \frac{1}{2m} \left[\left(\hat{P}_x - \frac{m\Omega y}{2} \right)^2 + \left(\hat{P}_y + \frac{m\Omega x}{2} \right)^2 \right] = \tag{145}$$

$$= \frac{1}{2m} \left[\hat{P}_x{}^2 + \hat{P}_y{}^2 \right] + \frac{m\Omega^2}{2} \left(x^2 + y^2 \right) + m\Omega \hat{L}_z,$$

where

$$\hat{L}_z = x\hat{P}_y - \hat{P}_x y \tag{146}$$

is the z-component of the angular momentum. (We do not consider a free motion along the field $\mathbf{B}$ which is trivial, so that $\hat{H}_t$ stands for the transverse part of the Hamiltonian.) Proceeding in the same fashion as in the section 2.5, we can introduce a couple of operators

$$\hat{a}_x = \frac{1}{\sqrt{2}} \left(q_x + \frac{d}{dq_x} \right), \quad \hat{a}_y = \frac{1}{\sqrt{2}} \left(q_y + \frac{d}{dq_y} \right) \tag{147}$$

with $q_x = x/L_H$ and $q_y = y/L_H$.

In terms of these operators the Hamiltonian (145) takes the form

$$\hat{H}_t = \hbar\Omega \left\{ \hat{a}_x^+ \hat{a}_x + \hat{a}_y^+ \hat{a}_y + 1 - i \left(\hat{a}_y \hat{a}_x^+ - \hat{a}_y^+ \hat{a}_x \right) \right\}. \qquad (148)$$

This quadratic form can be diagonalized with the help of the unitary transformation

$$\hat{a}_1 = \frac{1}{\sqrt{2}} \left(\hat{a}_x + i\hat{a}_y \right), \hat{a}_2 = \frac{1}{\sqrt{2}} \left(\hat{a}_x - i\hat{a}_y \right), \qquad (149)$$

which implies that both left and right hand side operators commute according to the Bose-like relations

$$\left[\hat{a}_i, \hat{a}_j^+ \right] = \delta_{ij}, [\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^+, \hat{a}_j^+ \right] = 0, (i, j = 1, 2, x, y). \qquad (150)$$

After the diagonalization, the operators $\hat{H}_t$ and $\hat{L}_z$ become

$$\hat{H}_t = 2\hbar\Omega \left\{ \hat{a}_1^+ \hat{a}_1 + \frac{1}{2} \right\}, \qquad (151)$$

$$\hat{L}_z = \hbar \left(\hat{a}_1^+ \hat{a}_1 - \hat{a}_2^+ \hat{a}_2 \right). \qquad (152)$$

These operators are linear combinations of the operator $\hat{n}_i = \hat{a}_i^+ \hat{a}_i$. On the other hand, from the commutation relations $[\hat{H}_t, \hat{L}_z] = [\hat{H}_t, \hat{n}_i] = [\hat{L}_z, \hat{n}_j] = 0$ it follows immediately that the eigenfunctions of the $\hat{n}_j$ are at the same time the eigenfunctions for the operators $\hat{H}_t$ and $\hat{L}_z$ with the corresponding eigenvalues

$$E_{n_1} = 2\hbar\Omega \left(n_1 + \frac{1}{2} \right), \qquad (153)$$

$$L_z = \hbar \left(n_1 - n_2 \right). \qquad (154)$$

We see therefore, that the wave function of a charged particle $\Psi_{n_1 n_2}$ is determined by a couple of quantum numbers n_1 and n_2. It may be obtained from the ground state Ψ_{00} in the same manner as we have done it in the section 2.5 for the case of the coherent state in the Landau gauge. The equations for the ground state generalizing the corresponding definition of the section 2.5 given by Eq. (74) yield $\hat{a}_1 \psi_{00} = 0$, $\hat{a}_2 \psi_{00} = 0$.

In the coordinate representation these equations read

$$L_H^2 \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{1}{2} (x + iy) \Psi_{00}(x, y) = 0, \qquad (155)$$

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$$-\left[L_H^2\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]+\frac{1}{2}\left(x-iy\right)\Psi_{00}(x,y)=0. \qquad (156)$$

Solving Eqs.(155), (156) and normalizing the solution, we found

$$\Psi_{00}(x,y)=\left(\frac{m\Omega}{\pi\hbar}\right)^{1/2}\exp\left[-\left(\frac{1}{2L_H^2}\right)\left(x^2+y^2\right)\right]. \qquad (157)$$

In full analogy with the equation (76) the wave function belonging to the energy $E_N=2\hbar\Omega(N+1/2)$ is given by

$$\Psi_{Nn}=\frac{\left(\hat{a}_1^+\right)^N}{\sqrt{N!}}\frac{\left(\hat{a}_2^+\right)^n}{\sqrt{n!}}\Psi_{00}. \qquad (158)$$

This wave function is degenerated with respect to the quantum number n which is an arbitrary integer determining the $L_z=\hbar(n-N)$. Introducing a complex coordinate $\rho=x+iy$ one can rewrite Eq. (158) in an explicit form

$$\Psi_{Nn}\left(\rho,\rho^*\right)=\left(\frac{2^{n-N}m\Omega}{\pi\hbar N!n!}\right)^{1/2}\left(\rho^*-\frac{\partial}{\partial\rho}\right)^N\rho^n e^{-|\rho|^2}. \qquad (159)$$

We determine then the coherent state $\Psi_{\alpha\zeta}$ as the eigenstate for the operators $\hat{a}_1$ and $\hat{a}_2$

$$\hat{a}_1\Psi_{\alpha\zeta}=\frac{\alpha}{L_H}\Psi_{\alpha\zeta}, \qquad (160)$$

$$\hat{a}_2\Psi_{\alpha\zeta}=\frac{\zeta}{L_H}\Psi_{\alpha\zeta}. \qquad (161)$$

In the coordinate representation these equations take form of the differential equations

$$\left(\frac{\partial}{\partial\rho^*}+\frac{1}{4L_H^2}\rho\right)\Psi_{\alpha\zeta}=\frac{\alpha}{2L_H^2}\Psi_{\alpha\zeta}, \qquad (162)$$

$$\left(\frac{\partial}{\partial\rho}+\frac{1}{4L_H^2}\rho^*\right)\Psi_{\alpha\zeta}=\frac{\zeta}{2L_H^2}\Psi_{\alpha\zeta}. \qquad (163)$$

The normalized solution of Eqs.(162) and (163), as one can check by the direct substitution, is

$$\Psi_{\alpha\zeta} = \frac{1}{L_H\sqrt{2\pi}}\exp\left[-\frac{1}{4L_H^2}\left(\mid\alpha\mid^2+\mid\zeta\mid^2\right)-\frac{1}{4L_H^2}\left(\rho-2\varsigma\right)\left(\rho^*-2\alpha\right)+\frac{\alpha\zeta}{2L_H^2}\right].$$
$$(164)$$

We see that $\Psi_{\alpha\zeta}$ is the Gauss-like wave function which means that $\Psi_{\alpha\zeta}$ minimize the uncertainty principle for the coordinate and momentum

$$\delta x_{\alpha\zeta}\delta p_{\alpha\zeta} = \frac{\hbar}{2}. \tag{165}$$

and, hence, is the coherent wave function in the sense of the Schrödinger definition.

7 Electron in the field of a thin solenoid. The Aharonov-Bohm effect

Consider an electron moving in the magnetic field of an infinitesimally thin and infinitely long solenoid whose magnetic field is oriented along the z-axis of the Cartesian coordinates. Let the flux through this solenoid be Φ, so that the magnetic field of the solenoid depends only on the coordinate $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ within the plane perpendicular to the magnetic field

$$\mathbf{B}\left(\mathbf{r}\right) = \Phi\delta\left(\mathbf{r}\right). \tag{166}$$

Since the Schrödinger equation depends on the vector potential $\mathbf{A}\left(\mathbf{r}\right)$, we have to find $\mathbf{A}\left(\mathbf{r}\right)$ such that

$$\mathrm{rot}\mathbf{A}\left(\mathbf{r}\right) = \Phi\delta\left(\mathbf{r}\right). \tag{167}$$

Equation (167) implies that

$$\oint \mathbf{A}\left(\mathbf{r}\right)d\mathbf{l} = \oint\oint \mathrm{rot}\mathbf{A}\left(\mathbf{r}\right)d\mathbf{S} = \Phi \tag{168}$$

for any contour around the singular point $\mathbf{r} = 0$.

It is easy to see that

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\Phi}{2\pi}\frac{\mathbf{k}\times\mathbf{r}}{r^2} = \frac{\Phi}{2\pi}\frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}. \tag{169}$$

satisfies the required conditions. ($\mathbf{i},\mathbf{j},\mathbf{k}$ are the unit vectors along the $X-, Y-$ and $Z-$ axes correspondingly).

We can introduce an angle $\theta\left(\mathbf{r}\right)$ between the $\mathbf{r}$ and X-axis by

$$\theta\left(\mathbf{r}\right) = \arctan\left(\frac{y}{x}\right). \tag{170}$$

In terms of $\theta\left(\mathbf{r}\right)$ the vector potential (169) reads

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\Phi}{2\pi}\frac{\partial\theta\left(\mathbf{r}\right)}{\partial\mathbf{r}}. \tag{171}$$

The Schrödinger equation for an electron in the field of the solenoid is given by

$$\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial\mathbf{r}} - \frac{e}{c}\mathbf{A}\left(\mathbf{r}\right)\right)^2 \Psi\left(\mathbf{r}\right) = E\Psi\left(\mathbf{r}\right) \tag{172}$$

with the vector potential determined by Eqs.(169) and (171).

In view of the axial symmetry of the problem the wave function should be periodic under rotations on the angle $\theta = 2\pi$ around the z-axis.

Consider a phase transformation of the wave function

$$\Psi'\left(\mathbf{r}\right) = e^{-i\frac{\Phi}{\Phi_0}\theta(\mathbf{r})}\Psi\left(\mathbf{r}\right), \tag{173}$$

where $\Phi_0 = \frac{2\pi\hbar c}{e}$ is the flux quantum. Since $\left|\Psi'\left(\mathbf{r}\right)\right|^2 = \left|\Psi\left(\mathbf{r}\right)\right|^2$ both $\Psi'\left(\mathbf{r}\right)$ and $\Psi\left(\mathbf{r}\right)$ belong to the same quantum state. On the other hand, the primed wave function satisfies the Schrödinger equation without vector potential

$$-\frac{\hbar}{2m_e}\frac{\partial^2}{\partial\mathbf{r}^2}\Psi'\left(\mathbf{r}\right) = E\Psi'\left(\mathbf{r}\right) \tag{174}$$

because of the relation

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial\mathbf{r}} - \frac{e}{c}\mathbf{A}\left(\mathbf{r}\right)\right)e^{i\frac{\Phi}{\Phi_0}\theta(\mathbf{r})}\Psi'\left(\mathbf{r}\right) = e^{i\frac{\Phi}{\Phi_0}\theta(\mathbf{r})}\left(\frac{\hbar}{i}\frac{\partial}{\partial\mathbf{r}} - \frac{e}{c}\mathbf{A}'\left(\mathbf{r}\right)\right)\Psi'\left(\mathbf{r}\right) \tag{175}$$

in which

$$\mathbf{A}'\left(\mathbf{r}\right) = \mathbf{A}\left(\mathbf{r}\right) - \frac{\Phi}{2\pi}\frac{\partial\theta\left(\mathbf{r}\right)}{\partial\mathbf{r}} = 0 \tag{176}$$

in view of Eq. (171).

We must supply Eq. (174) with the boundary conditions under rotation which are rather unusual and depending on the flux Φ. Indeed, since $\Psi\left(\mathbf{r}\right)$ is the 2π-periodic, we have from Eq. (173)

$$\Psi'\left(\theta + 2\pi\right) = e^{-i2\pi\frac{\Phi}{\Phi_0}}\Psi'\left(\theta\right). \tag{177}$$

In the polar coordinates r and θ Eq. (174) reads

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\right\}\Psi'(r,\theta)=E\Psi'(r,\theta) \qquad (178)$$

and one can separate variables in $\Psi(\mathbf{r})\equiv\Psi(r,\theta)$

$$\Psi(r,\theta)=f(r)\,u_m(\theta), \qquad (179)$$

where $u_m(\theta)$ is the eigenfunction of the operator $\hat{L}_z=\frac{\hbar}{i}\frac{\partial}{\partial\theta}$ because of the axial symmetry of the problem in question

$$u_m(\theta)=\frac{1}{\sqrt{2\pi}}e^{im\theta}. \qquad (180)$$

Note first that in the polar coordinates $x=r\cos\theta, y=r\sin\theta$ and the angle θ in Eq. (170), in fact, does not depend on coordinates.

Thus, the primed wavefunction becomes

$$\Psi'(r,\theta)=f(r)\frac{e^{i\left(m-\frac{\Phi}{\Phi_0}\right)\theta}}{\sqrt{2\pi}}, \qquad (181)$$

where m is an integer (the L_z quantum number).

We see therefore that one can interpret the angular dependence of the wavefunction (181) in terms of the angular momentum projection on the solenoid axis, i.e.

$$\frac{1}{\sqrt{2\pi}}e^{i\frac{L_z\theta}{\hbar}}=\frac{1}{\sqrt{2\pi}}e^{i\left(m-\frac{\Phi}{\Phi_0}\right)\theta}. \qquad (182)$$

This implies that the angular momentum is now depends on the flux through the solenoid:

$$L_z=\hbar\left(m-\frac{\Phi}{\Phi_0}\right). \qquad (183)$$

Substituting then $\Psi'(r,\theta)$ (181) into Eq. (178) we arrive at the equation for the radial wavefunction $f(r)$:

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}+\frac{\left(m-\frac{\Phi}{\Phi_0}\right)^2}{r^2}f-\frac{2m_eE}{\hbar^2}f=0. \qquad (184)$$

This is the Bessel equation and its regular at $\mathbf{r}=0$ solution is given by the Bessel function J_ν:

$$f_\nu\left(r\right) = CJ_\nu\left(kr\right), \tag{185}$$

where

$$k = \sqrt{\frac{2m_eE}{\hbar^2}} \ \ and \ \nu = \left|m - \frac{\Phi}{\Phi_0}\right|. \tag{186}$$

In case when there is an unpenetrable cylinder at $r = R$, we have $f_\nu\left(R\right) = 0$ and correspondingly

$$J_\nu\left(kR\right) = 0, \tag{187}$$

which means that kR is one of the roots of the Bessel function $\kappa\left(n,\nu\right)$

$$k_{n\nu} = \frac{\kappa\left(n,\nu\right)}{R}, \tag{188}$$

so that the energy spectrum is a discrete and determined by equations (186) and (188)

$$E\left(n,\nu\right) = \frac{\hbar^2}{2mR^2}\kappa^2\left(n,\nu\right). \tag{189}$$

We see that angular momentum L_z (183) and the energy spectrum (189) depend on the magnetic field Φ/Φ_0 though electron moves in the region $r > 0$, where magnetic field equals zero. This paradoxical phenomenon which has no analog in classical electrodynamics was predicted in 1959 by Ahronov and Bohm. In classical mechanics the Lorentz force acts locally and therefore has no impact on a charged particle in the region where $B = 0$ even though the vector potential is nonzero. After its theoretical prediction in 1959 the Ahronov-Bohm effect have been found then experimentally and has numerous manifestations in the modern physics.

8 The density matrix of a charged particle in quantizing magnetic field

So far we have considered the Landau problem within the Schrödinger equation approach which implies that the charged particle is isolated from the environment and only under this assumption a description in terms of the wave function is relevant. In reality electrons in the solids are involved in different interactions and move under the action of the atomic forces from the crystal lattice. Another words they correlate somehow with the rest

of the sample. This correlation means that the true quantum mechanical description should be based not on the wave function ψ but rather must be done in terms of the density matrix $\hat{\rho}$. The density matrix approach is an alternative to the Schrödinger equation description in case when a system is in contact with the environment. In this section we will give a description of the Landau problem in terms of the density matrix in the most simple case which assumes a contact between the charged particle and the thermostat (environment) being at the temperature T.

8.1 The density matrix in the Landau problem

The Landau energy spectrum and the wave functions have been calculated in detail in section 2.4. According to the results of this section we can write the density matrix $\hat{\rho}\left(\mathbf{r}, \mathbf{r}', \beta\right)$ in the Landau basis (taken in the Landau gauge) as follows:

$$\hat{\rho}\left(\mathbf{r}, \mathbf{r}', \beta\right) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp_y \rho_\perp(q, q', \beta) e^{-i\frac{p_y}{\hbar}(y-y')} \rho_\parallel(z - z', \beta). \qquad (190)$$

Here the quantity $\rho_\parallel(z - z', \beta)$ stands for the longitudinal density matrix for a free particle moving parallel to the magnetic field

$$\rho_\parallel(z - z', \beta) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp_z e^{-\beta \frac{p_z^2}{2m} - i\frac{p_z}{\hbar}(z-z')} \qquad (191)$$

and the function $\rho_\perp(q, q', \beta)$ is the density matrix of the Larmor oscillator centered at the coordinate $x_0(p_y) = -cp_y/eB$ in the plane perpendicular to the applied magnetic field:

$$\rho_\perp(q, q', \beta) = \sum_{n=0}^{\infty} e^{-\beta\hbar\Omega\left(n+\frac{1}{2}\right)} \varphi_n(q) \varphi_n(q'). \qquad (192)$$

The Landau basis $\varphi_n(q)$ is given by the Eq. (81), and dimensionless coordinates q and q' are connected with the $x-$axes coordinates by the relations

$$L_H q = x - x_0(p_y), \quad L_H q' = x' - x_0(p_y). \qquad (193)$$

Completing integration in the Eq. (191) we obtain

$$\rho_\parallel(z - z', \beta) = \sqrt{\frac{m}{2\pi\beta\hbar^2}} e^{-\frac{m}{2\beta\hbar^2}(z-z')^2}. \qquad (194)$$

This quantity, normalized by the condition $\rho_\parallel(z - z', 0) = \delta(z - z')$, is exactly the statistical operator for a free particle in a one dimensional space.

To calculate the $\rho_\perp(q, q', \beta)$ is a more tricky business. With this purpose in mind, we first derive the differential equation for this quantity. To do this, note that from the definition of the operators $\hat{a}$ and $\hat{a}^+$ by Eq. (70) and Eqs.(84), (85) it follows that

$$q\varphi_n(q) = \frac{1}{\sqrt{2}} \left(\sqrt{n}\varphi_{n-1}(q) + \sqrt{n+1}\varphi_{n+1}(q) \right), \tag{195}$$

$$\frac{\partial}{\partial q}\varphi_n(q) = \frac{1}{\sqrt{2}} \left(\sqrt{n}\varphi_{n-1}(q) - \sqrt{n+1}\varphi_{n+1}(q) \right). \tag{196}$$

Using these equations as well as Eq. (192), we have

$$\frac{\partial}{\partial q}\rho_\perp(q, q', \beta) = e^{-\beta\hbar\Omega}f(q, q') - f(q', q), \tag{197}$$

where the following function was introduced

$$f(q', q) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\Omega(n+1/2)} \sqrt{n+1}\varphi_n(q')\varphi_{n+1}(q). \tag{198}$$

With the help of Eqs.(195) and (192) we find a useful relations between the function $f(q', q)$ and the perpendicular component of the statistical operator:

$$q\rho_\perp(q, q', \beta) = e^{-\beta\hbar\Omega}f(q, q') + f(q', q), \tag{199}$$

$$q'\rho_\perp(q, q', \beta) = e^{-\beta\hbar\Omega}f(q', q) + f(q, q'). \tag{200}$$

Combining Eqs.(197)-(200) we arrive at the differential equation for the density matrix $\rho_\perp$, which reads

$$\frac{\partial}{\partial q}\rho_\perp(q, q', \beta) = \left(-\frac{q}{\tanh\beta\hbar\Omega} + \frac{q'}{\sinh\beta\hbar\Omega} \right)\rho_\perp(q, q', \beta). \tag{201}$$

The solution of this simple equation is trivial an yields

$$\rho_\perp(q, q', \beta) = C(q', \beta)\exp\left[-\left(\frac{q^2}{2\tanh\beta\hbar\Omega} - \frac{qq'}{\sinh\beta\hbar\Omega} \right) \right]. \tag{202}$$

According to the definition (192) the quantity $\rho_\perp(q, q', \beta)$ is symmetric with respect to the substitution $q \to q'$. This condition tells us that the constant $C(q', \beta)$ should be taken in the form

$$C(q', \beta) = C_0(\beta) \exp\left[-\left(\frac{q'^2}{2\tanh\beta\hbar\Omega}\right)\right].\tag{203}$$

It follows also from the Eq. (65) that the function $\rho_\perp(q, q', 0)$ should be normalized by the condition

$$\rho_\perp(q, q', 0) = \delta\left(q - q'\right).\tag{204}$$

Choosing then the constant $C_0(\beta)$ to satisfy the equation (4.28), we have

$$\rho_\perp(q, q', \beta) = (2\pi\sinh\beta\hbar\Omega)^{-1/2} \exp\left[-\frac{(q^2 + q'^2)}{2\tanh\beta\hbar\Omega} + \frac{qq'}{\sinh\beta\hbar\Omega}\right].\tag{205}$$

Substituting the Eq. (205) into the Eq. (190), we find

$$\hat{\rho}\left(\mathbf{r}, \mathbf{r}', \beta\right) = \rho_\perp(x, x', y, y', \beta)\rho_\parallel(z - z', \beta),\tag{206}$$

where the perpendicular component of the density matrix
$\rho_\perp(x, x', y, y', \beta) = \rho_\perp(x, x', y - y', \beta)$ is determined by the integral

$$\rho_\perp(x, x', y, y', \beta) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp_y \rho_\perp(q, q') e^{-i\frac{p_y}{\hbar}(y - y')}\tag{207}$$

and the dependence on the momentum p_y enters the function $\rho_\perp(q, q')$ through the dimensionless coordinates

$$q(p_y) = [x - x_0(p_y)]/L_H, \quad q'(p_y) = [x' - x_0(p_y)]/L_H.\tag{208}$$

We can single out of the Eq. (207) $\rho_{osc}(x, x', \beta)$ the statistical operator of the quantum oscillator of the frequency Ω so that $\rho_\perp(x, x', y, y', \beta)$ can be written as a product

$$\rho_\perp(x, x', y, y', \beta) = \rho_{osc}(x, x', \beta)G(x, x', y, y', \beta),\tag{209}$$

where $\rho_{osc}(x, x', \beta)$ is given by the formula

$$\rho_{osc}(x, x', \beta) = \left(\frac{m\Omega}{2\pi\hbar\sinh\beta\hbar\Omega}\right)^{1/2} \exp\left[-\left(\frac{m\Omega}{2\hbar}\right)\left(\frac{x^2 + x'^2}{\tanh\beta\hbar\Omega} - \frac{2xx'}{\sinh\beta\hbar\Omega}\right)\right].\tag{210}$$

The G function is given by the Gauss integral

$$G(x, x', y, y', \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_y e^{-Ak_y^2 + Bk_y} = \frac{1}{2\pi} \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}} \qquad (211)$$

with

$$A = L_H^2 \tanh\left(\frac{\beta\hbar\Omega}{2}\right), B(x, x', y, y', \beta) = \tanh\left(\frac{\beta\hbar\Omega}{2}\right)(x + x') + i(y - y'). \qquad (212)$$

Combining all these equations , we finally have

$$\hat{\rho}\left(\mathbf{r}, \mathbf{r}', \beta\right) = \frac{\rho_{\parallel}(z - z', \beta)}{4\pi L_H^2 \sinh\left(\frac{\beta\hbar\Omega}{2}\right)} e^{-S(x, x', y, y', \beta)}, \qquad (213)$$

where

$$S(x, x', y, y', \beta) = \qquad (214)$$
$$= \frac{1}{4L_H^2} \left\{ \coth\left(\frac{\beta\hbar\Omega}{2}\right) \left[(x - x')^2 + (y - y')^2\right] + 2i(x + x')(y - y') \right\}.$$

The partition function of the problem in question is given by

$$Q\left(\beta\right) = \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz \hat{\rho}\left(\mathbf{r}, \mathbf{r}, \beta\right). \qquad (215)$$

Taking into account that $S(x, x, y, y, \beta) \equiv 0$ we see that $\hat{\rho}(\mathbf{r}, \mathbf{r}, \beta)$ does not depend on the coordinate $\mathbf{r}$

$$\hat{\rho}(\mathbf{r}, \mathbf{r}, \beta) = \frac{\rho_{\parallel}(0, \beta)}{4\pi L_H^2 \sinh\left(\frac{\beta\hbar\Omega}{2}\right)}. \qquad (216)$$

Then, after the trivial integration in the Eq. (215) we obtain an explicit formula for the partition function $Q\left(\beta\right)$:

$$Q(\beta) = \frac{\Phi}{\Phi_0} \frac{1}{2\sinh\left(\frac{\beta\hbar\Omega}{2}\right)} L_z \sqrt{\frac{m}{2\pi\hbar^2\beta}}. \qquad (217)$$

The origin of each factor in the equation (217) is absolutely clear: $g = L_x L_y / 2\pi L_H^2 = \Phi/\Phi_0$ is the degeneracy of the Landau level on the Larmor orbit center position, $\left[2\sinh\left(\frac{\beta\hbar\Omega}{2}\right)\right]^{-1}$ is the partition function of

the quantum oscillator of the cyclotron frequency Ω, and the last factor $L_z(m/2\pi\hbar^2\beta)^{1/2}$ is the partition function of a free particle in one dimension associated with its motion along the z axis (i.e. along the magnetic field).

The free energy $F = -(1/\beta)\ln Q(\beta)$ is given by

$$F = \frac{\hbar\Omega}{2} + T\ln\left(1 - e^{-\frac{\hbar\Omega}{T}}\right) - T\ln\left(\frac{\Phi}{\Phi_0}L_z\sqrt{\frac{mT}{2\pi\hbar^2}}\right). \qquad (218)$$

The sum of the first two terms in the Eq. (218) is exactly the free energy of the oscillator with the frequency Ω, whereas the last term is due to the degeneracy of the Landau orbits and because of the free motion of a particle along the magnetic field.

9 The Green's function of a particle in external magnetic field

The results of the previous section, as we will show, may be used for the calculations of the Green's function of the Landau problem because of the formal similarity of the equation of motion in both cases. We start from the equation of motion for the Green's function which in a general form reads as follows

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}(\mathbf{r})\right) G\left(\mathbf{r}, t, \mathbf{r}', t'\right) = i\hbar\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \qquad (219)$$

where $\hat{H}(\mathbf{r})$ is the Hamiltonian of the system.

If the eigenvalue equation is solved

$$\hat{H}(\mathbf{r})\Psi_n(\mathbf{r}) = E_n\Psi_n(\mathbf{r}) \qquad (220)$$

so that the energy spectrum E_n and the wave functions $\Psi_n(\mathbf{r})$ are found explicitly, then it is straightforward to check that the Green's function can be calculated as a sum over the quantum spectrum:

$$G\left(\mathbf{r}, t, \mathbf{r}', t'\right) = \Theta(t - t')\sum_n e^{-\frac{i}{\hbar}E_n(t-t')}\Psi_n(\mathbf{r})\Psi_n^*(\mathbf{r}'), \qquad (221)$$

where

$$\Theta\left(\tau\right) = \begin{cases} 1, & if\,\tau \geq 0 \\ 0, & if\,\tau < 0 \end{cases}$$

is the Heavyside step-function.

Putting $t' = 0$ in Eq. (221) and compare it with equation, describing the coordinate representation for the statistical operator $\hat{\rho}(\mathbf{r}, \mathbf{r}', \beta)$, we found a simple relation between the Green's function and the density matrix:

$$G\left(\mathbf{r}, t, \mathbf{r}', 0\right) = \hat{\rho}\left(\mathbf{r}, \mathbf{r}', \beta\right)|_{\beta = \frac{it}{\hbar}}. \tag{222}$$

Since we have calculated above the density matrix for the Landau problem, the Green's function for a charged particle in the magnetic field follows immediately from Eqs.(213) and (194)

$$G\left(\mathbf{r}, t, \mathbf{r}', 0\right) = G_{\parallel}\left(z - z', t\right) G_{\perp}\left(\boldsymbol{\rho}, t, \boldsymbol{\rho}', 0\right). \tag{223}$$

Here $G_{\parallel}$ is the Green's function of a free particle moving along the z-axis (i.e. along the magnetic field $\mathbf{B}$) $G_{\parallel}\left(z - z', t\right) = \rho_{\parallel}\left(z - z', \beta\right)|_{\beta = \frac{it}{\hbar}}$,

$$G_{\parallel}\left(z - z', t\right) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \exp\left[\frac{im}{2\hbar t}(z - z')^2\right] \tag{224}$$

and $G_{\perp}$ stands for the Green's function of a charged particle moving within the plane perpendicular quantizing magnetic field

$$G_{\perp}\left(\boldsymbol{\rho}, \boldsymbol{\rho}', t\right) = \frac{1}{4\pi i L_H^2 \sin\left(\frac{\Omega t}{2}\right)} e^{i\tilde{S}(x, x', y, y', t)}, \tag{225}$$

where

$$\tilde{S} = \frac{1}{4L_H^2}\left\{\cot\left(\frac{\Omega t}{2}\right)\left[(x - x')^2 + (y - y')^2\right] + 2\left(x + x'\right)^2 \left(y - y'\right)^2\right\}. \tag{226}$$

The above equations for the Green's function $G\left(\mathbf{r}, t, \mathbf{r}', 0\right)$ have been obtained in the Landau gauge $\mathbf{A} = (0, By, 0)$. A natural question arises in this connection how the gauge transformations may influence the shape of the Green's function determined by Eq. (221). To answer this question let us rewrite the Hamiltonian in the eigenvalue equation (220) in the following form

$$\hat{H} = \frac{1}{2m}\left(\mathbf{D_A}\right)^2, \tag{227}$$

where

$$\mathbf{D_A} = (\frac{\hbar}{i}\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}). \tag{228}$$

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Consider now the gauge $\varphi(\mathbf{r})$ transformations given by two simultaneous relations: $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f(\mathbf{r})$ and $\Psi'_n(\mathbf{r}) = \Psi_n(\mathbf{r})e^{i\varphi(\mathbf{r})}$. If the phase $\varphi(\mathbf{r})$ in these transformations satisfies the condition $\hbar\boldsymbol{\nabla}\varphi(\mathbf{r}) = \boldsymbol{\nabla} f(\mathbf{r})e/c$ it is straightforward to see that a following equation holds

$$\mathbf{D}_{\mathbf{A}'}\Psi'_n(\mathbf{r}) = e^{i\varphi(\mathbf{r})}\mathbf{D}_{\mathbf{A}}\Psi_n(\mathbf{r}). \tag{229}$$

The latter means that changes in the vector potential due to the gradient term $\boldsymbol{\nabla} f(\mathbf{r})$ may be compensated by the gauge transformation $\Psi'_n(\mathbf{r}) = \Psi_n(\mathbf{r})e^{i\varphi(\mathbf{r})}$ with $\varphi(\mathbf{r}) = \frac{2\pi}{\Phi_0}f(\mathbf{r}) + C$ and the theory became gauge invariant under these transformations. Since the function $\Psi'_n(\mathbf{r})$ is the eigenfunction of the Schrödinger equation (220) belonging to the same eigenvalue E_n, as the wave function $\Psi_n(\mathbf{r})$, the Green's function (221) under the gauge transformation $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f(\mathbf{r})$ acquire an additional factor g:

$$g = \exp\left[\frac{2\pi}{\Phi_0}i\left(f(\mathbf{r}) - f(\mathbf{r}')\right)\right]. \tag{230}$$

In particular case of the symmetric gauge $\mathbf{A} = \frac{1}{2}[\mathbf{Br}]$ the function $f(\mathbf{r})$ should be tacking in the form $f = \frac{1}{2}Bxy$, so that the gauge factor is given by

$$g = \exp\left[\frac{\pi B}{\Phi_0}i\left(xy - x'y'\right)\right]. \tag{231}$$

10   The supersymmetry of the Landau problem

The supersymmetry of a system as the invariance of its Hamiltonian under the transformations of bosons into fermions and vice versa has been considered first in the quantum field theory. This notion appeared to be extremely creative both from physical and mathematical points of view. For the first time a matter (fermions) and carriers of interactions (bosons) have been involved into a theory on the equal footing. It was novel also that commuting and anticommuting variables have been incorporated into a new type of mathematics - the superalgebra. The basic property of the supersymmetry is that it unities in a nontrivial way the continuous and discrete transformations. Except the quantum field theory the ideas and methods of the supersymmetry have been spread wide over the different branches of physics: the statistical physics, the nuclear physics, the quantum mechanics and so on.

In this section we will show that incorporation of the discrete spin variable into the Landau problem makes the latter belonging to the so called supersymmetric quantum mechanics. Because the supersymmetric quantum mechanics so far is not a common textbook knowledge, we have to consider first some fundamentals of the supersymmetry in the nonrelativistic quantum mechanics. After that we will go ahead with the consideration of the supersymmetry in the Landau problem.

The simplest way to introduce operators transforming bosons into fermions and vice versa is as follows

$$\hat{Q}_+ \mid N_B, N_F\rangle \propto \mid N_B - 1, N_F + 1\rangle, \qquad (232)$$

$$\hat{Q}_- \mid N_B, N_F\rangle \propto \mid N_B + 1, N_F - 1\rangle, \qquad (233)$$

$\mid N_B, N_F\rangle$ is the state vector with fixed number of bosons N_B and fermions N_F. The integers N_B and N_F can take the following values: $N_B = 0, 1, 2...,$ but N_F takes only two values $N_F = 0, 1$.

The operator $\hat{Q}_+$ transforms bosons into fermions (i.e. annihilates the boson and creates the fermion) and $\hat{Q}_-$ contrary destroys one fermion and creates one boson. These operators may be presented in an evident form with the help of the creation and annihilation operators

$$\hat{Q}_+ = q\hat{b}\hat{f}^+, \quad \hat{Q}_- = q\hat{b}^+\hat{f}, \qquad (234)$$

where $\hat{b}$ and $\hat{f}$ satisfy standard for bosons and fermions commutation rules:

$$\left[\hat{b}, \hat{b}^+\right] = 1, \quad \left\{\hat{f}, \hat{f}^+\right\} \equiv \hat{f}\hat{f}^+ + \hat{f}^+\hat{f} = 1, \quad \hat{f}^2 = \hat{f}^{+2} = 0, \quad \left[\hat{b}, \hat{f}\right] = 0. \qquad (235)$$

The nilpotentcy of the fermion operators (i.e. the property $\hat{f}^2 = \hat{f}^{+2} = 0$) makes the operators $\hat{Q}_\pm$ nilpotent too

$$\hat{Q}_+^2 = \hat{Q}_-^2 = 0. \qquad (236)$$

This property is closely related with the anticommutation. Let us introduce two Hermitian operators $\hat{Q}_1$ and $\hat{Q}_2$ by the relations

$$\hat{Q}_1 = \hat{Q}_+ + \hat{Q}_-, \qquad \hat{Q}_2 = -i\left(\hat{Q}_+ - \hat{Q}_-\right). \qquad (237)$$

It is easy to check that these operators are anticommuting

$$\left\{ \hat{Q}_1, \hat{Q}_2 \right\} = 0 \tag{238}$$

and their squares satisfy the following equations:

$$\hat{Q}_1^2 = \hat{Q}_2^2 = \left\{ \hat{Q}_+, \hat{Q}_- \right\}. \tag{239}$$

These equations prompt us the simplest form for the Hamiltonian $\hat{H}$, possessing the supersymmetry, i.e. the one which is invariant under the transformations given by Eq. (232) and (233) and mixing bosons with fermions:

$$\hat{H} = \hat{Q}_1^2 = \hat{Q}_2^2 = \left\{ \hat{Q}_+, \hat{Q}_- \right\}. \tag{240}$$

The supersymmetry of the Hamiltonian (240) means that it does commute with any operator $\hat{Q}_\alpha$ (where $\alpha = \pm$ or $1,2$), so that for any $\hat{Q}_\alpha$ holds

$$\left[\hat{H}, \hat{Q}_\alpha \right] = 0. \tag{241}$$

Two important properties concerning the energy spectrum follows immediately from the definition of the supersymmetric Hamiltonian of Eq. (240). First, the energy spectrum given by the eigen equation

$$\hat{H}\Psi = E\Psi \tag{242}$$

is nonnegative $E \geq 0$ since $\hat{H}$ is determined as the square of the Hermitian operator. Second, the energy levels with nonzero energies $E \neq 0$ are degenerated twice. These statements may be proved as follows. Owing to the commutation relation of Eq. (241), the Hamiltonian $\hat{H}$ and the operators $\hat{Q}_1$ or $\hat{Q}_2$ should have a common set of eigenvectors. From the above equations and definitions it is straightforward to see that the eigenvector of the operator $\hat{Q}_1$ is at the same time the eigenvector of the Hamiltonian $\hat{H}$:

$$\hat{Q}_1 \Psi_1 = q\Psi_1, \quad \hat{H}\Psi_1 = E\Psi_1 = q^2 \Psi_1. \tag{243}$$

Let us define the vector Ψ_2 by the relation

$$\Psi_2 = \hat{Q}_2 \Psi_1. \tag{244}$$

It follows then that Ψ_2 is the eigenvector of $\hat{Q}_1$ with the eigenvalue $-q$:

$$\hat{Q}_1 \Psi_2 = \hat{Q}_1 \hat{Q}_2 \Psi_1 = -\hat{Q}_2 \hat{Q}_1 \Psi_1 = -q\Psi_2. \tag{245}$$

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On the other hand, since $\left[\hat{H}, \hat{Q}_2\right] = 0$, we obtain

$$\hat{H}\Psi_2 = \hat{H}\hat{Q}_2\Psi_1 = \hat{Q}_2\hat{H}\Psi_1 = q^2\Psi_2. \tag{246}$$

Thus, if $q \neq 0$ then both Ψ_1 and Ψ_2 belong to the eigenvalue $E = q^2$ which means a double degeneracy, whereas the energy level $E = 0$ ($q = 0$) is nondegenerated. This properties are the direct consequence of the supersymmetry of the Hamiltonian (240).

It is instructive for further consideration to express $\hat{H}$ in terms of the operators $\hat{b}$ and $\hat{f}$. Using Eqs.(240) and (234) we have

$$\hat{H} = \left\{\hat{Q}_+, \hat{Q}_-\right\} = \hat{H}_B + \hat{H}_F. \tag{247}$$

Therefore, we see that $\hat{H}$ is a sum of two Hamiltonians quadric in operators, which we will call the bosonic and fermionic oscillators:

$$\hat{H}_B = q^2 \left(\hat{b}^+\hat{b} + \frac{1}{2}\right), E_B = q^2 \left(N_B + \frac{1}{2}\right), N_B = 0, 1, 2..., \tag{248}$$

$$\hat{H}_F = q^2 \left(\hat{f}^+\hat{f} - \frac{1}{2}\right), E_F = q^2 \left(N_F - \frac{1}{2}\right), N_F = 0, 1. \tag{249}$$

The frequencies of these two oscillators are the same $\omega = q^2$ which makes the Hamiltonian of Eq. (240) supersymmetric.

The eigenvalues of the Hamiltonian $\hat{H}$ are positive and given by the sum

$$E_{N_B N_F} = E_B + E_F = \omega \left(N_B + N_F\right). \tag{250}$$

The ground state (the vacuum) of the Hamiltonian $\hat{H}$ corresponds to the quantum numbers $N_B = N_F = 0$, so that the positive energy of the boson oscillator ground state, $E_B^0 = \omega/2$, is exactly compensated by the negative fermion vacuum energy $E_F^0 = -\omega/2$. This is the simplest manifestation of the famous cancellation of the vacuum zero oscillations energy in the supersymmetric theories.

In the quantum field theory, owing to the infinite degrees of freedom, the energies of fermionic and bosonic vacua are infinite and have opposite signs. Thus, the problems with the infinite vacuum energies in nonsupersymmetric theories are no more than an artifact arising because of the inappropriate division of the zero vacuum energy of the "unified theory" including bosons and fermions into two (infinite) parts: positive bosonic and negative fermionic. We see that in the supersymmetric theory the fermionic and bosonic vacuum energies simply cancel each other.

The $\hat{Q}_{\pm}$ operators may be generalized in a way preserving the supersymmetric form of the Hamiltonian given by the Eq. (240). For example, if we take them in the form

$$\hat{Q}_{+} = \hat{B}\left(\hat{b}, \hat{b}^{+}\right) \hat{f}^{+}, \tag{251}$$

$$\hat{Q}_{-} = \hat{B}^{+}\left(\hat{b}, \hat{b}^{+}\right) \hat{f}, \tag{252}$$

then it is straightforward to check that the key relation $\left[\hat{H}, \hat{Q}_{\pm}\right] = 0$ holds for an arbitrary function $\hat{B}\left(\hat{b}, \hat{b}\right)$ of the bosonic operators $\hat{b}$ and $\hat{b}^{+}$, because of the nilpotentcy of the operators (251) and (252). On the other hand, the supersymmetric Hamiltonian (247) after substitution of the operators (251) and (252) describes a system of bosons interacting with themselves and fermions, in contrast to the noninteracting fermionic and bosonic oscillators in case when operators $\hat{Q}_{\pm}$ are determined by the Eq. (234).

In as much as the fermion filling number N_F may take only two values $N_F = 0, 1$ it is convenient to use a two-component wave vector in the form

$$\Psi = \left(\begin{array}{c} \Psi_1 \\ \Psi_0 \end{array}\right) \tag{253}$$

with Ψ_1 corresponding to $N_F = 1$, and Ψ_0 to $N_F = 0$. The Fermi operators $\hat{f}$ and $\hat{f}^{+}$ in this representation are given by the 2×2 matrix

$$\hat{f}^{+} = \hat{\sigma}^{+} = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \quad \hat{f} = \hat{\sigma}^{-} = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right), \tag{254}$$

where $\hat{\sigma}^{\pm} = 1/2(\hat{\sigma}_1 \pm i\hat{\sigma}_2)$ and symbol $\hat{\sigma}_j (j = 1, 2, 3)$ stands for the Pauli matrices.

The Hamiltonian $\hat{H}$ in the matrix representation takes the form

$$\hat{H} = \left\{\hat{Q}_{+}, \hat{Q}_{-}\right\} = \frac{1}{2}\left\{\hat{B}, \hat{B}^{+}\right\} + \frac{1}{2}\left[\hat{B}, \hat{B}^{+}\right]\hat{\sigma}_3. \tag{255}$$

We see that fermionic degree of the freedom (given by the term containing $\hat{\sigma}_3$) vanishes if the commutator $\left[\hat{B}, \hat{B}^{+}\right] = 0$.

Taking then $\hat{B}$ in the form

$$\hat{B} = \frac{1}{\sqrt{2}}\left[i\hat{P} + W(x)\right], \quad \hat{P} = \frac{\hbar}{i}\frac{d}{dx} \tag{256}$$

where $W(x)$ is an arbitrary function of the coordinate, we have

$$\hat{H} = \frac{1}{2}\left[\hat{P}^2 + W^2(x) + \hat{\sigma}_3 + \hbar\frac{dW(x)}{dx}\right]. \tag{257}$$

This differential operator is known as the Hamiltonian of the super-symmetric quantum mechanics of Witten. It takes the form of the Pauli Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2} + U(x) + \hat{\sigma}_3\mu_0 B(x), \tag{258}$$

if we adopt the potential energy to be $U(x) = W^2(x)/2$ and associate the Zeeman splitting $\pm\mu_0 B(x)$ with the last term in the Eq. (258) i.e. choose the "magnetic field " according to the relation $\hbar dW(x)/dx = \mu_0 B(x)$. Of course, it is not true magnetic field since there are no real magnetic field in one dimension.

In the three dimensional case the Pauli Hamiltonian reads

$$\hat{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + U - \mu_0 B\hat{s}, \quad \hat{s} = \frac{\hbar}{2}\hat{\sigma}_3. \tag{259}$$

The eigenfunctions of this Hamiltonian can be written as a product of the coordinate and spin functions

$$\Psi_{n,P_z,\nu,s_z} = \Psi_{nP_z\nu}(\mathbf{r})\chi\left(s_z\right). \tag{260}$$

The index ν here depends on the momentum P_y which determines the Larmore orbit position in the Landau gauge ($\mathbf{A} = (0, Bx, 0)$), or on the integer m in the case of symmetric gauge ($\mathbf{A} = 1/2(-By, Bx, 0)$).

The energy spectrum for $U \equiv 0$ is given by the equations

$$E_{nP_z}(s_z) = E_n(s_z) + \frac{P_z^2}{2m}, \tag{261}$$

$$E_n(s_z) = \hbar\Omega\left(n + \frac{1}{2}\right) - 2\mu_0 B s_z. \tag{262}$$

We see that the transverse energy (262) possesses the properties of the supersymmetric quantum mechanics. It becomes clear if we rewrite Eq. (262) in the form

$$E_n(s_z) \equiv E_N = \hbar\Omega N, \tag{263}$$

where $N = n + s_z + 1/2$ is a sum of the two quantum numbers.

The ground state, $N = 0$, $(n = 0, s_z = -1/2)$ has zero energy $E_0 = 0$ and this energy level is not degenerated. Contrary, the levels with $N \neq 0$ are degenerated twice since two states with different quantum numbers $n = N, s_z = -1/2$ and $n = N - 1, s_z = 1/2$ belong to the same energy level. This is exactly what we should have in the supersymmetric quantum mechanics.

(We do not consider here the degeneracy on the orbit centre position). In essence, the supersymmetry of the energy spectrum $E_n(s_z)$ stems from the fact that the Bohr magneton equals to $\mu_0 = e\hbar/2mc$ so that the energy $\mu_0 B$ is exactly one half of the cyclotron energy $\hbar\Omega$. Because of that, the transverse part of the Hamiltonian (259) can be written in the supersymmetric form

$$\hat{H}_\perp = \frac{1}{2m}\left(\hat{\mathbf{p}}_\perp - \frac{e}{c}\mathbf{A}_\perp\right)^2 - \mu_0 H \hat{s} = \hbar\Omega\left(\hat{b}^+\hat{b} + \frac{1}{2}\right) + \hbar\Omega\left(\hat{f}^+\hat{f} - \frac{1}{2}\right),$$
(264)

where the Fermi operators $\hat{f}^+$ and $\hat{f}$ are taken in the form (254) while the Bose operators $\hat{b}^+$ and $\hat{b}$ are given by

$$\hat{b} = (\hat{\pi}_y + i\hat{\pi}_x)\,(2m\hbar)^{-1/2}\,, \quad \left[\hat{b}, \hat{b}^+\right] = 1$$
(265)

with $\hat{\pi} = \left(\hat{\mathbf{p}}_\perp - \frac{e}{c}\mathbf{A}_\perp\right)$.

The orbital moment corresponds here the bosonic degree of freedom with the quantum numbers $N_B = n(0, 1, 2...)$ whereas the spin variable plays the role of the fermionic degree of freedom with $N_F = s_z + 1/2$. The energy spectrum of the Hamiltonian (264) is equal to

$$E(N_B, N_F) = \hbar\Omega(N_B + N_F).$$
(266)

The above consideration shows that the supersymmetry of the electron moving in an external magnetic field is not an abstract mathematical construction, since it has a practical realization in the Landau problem.

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