Thermodynamics of two-dimensional electrons on Landau levels

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Abstract

We review here some aspects of modern thermodynamics of different two-dimensional electron systems (2DES) on Landau levels (i.e. in quantizing magnetic field). Some ideas are illustrated in the free electron 2DES model and are applied to calculation of the two dimensional de Haas- van Alphen effect. The main player in this game are the sharp quantum oscillations of the chemical potential, $\mu(B,T)$, which define the magnetic oscillations of the all relevant physical properties.

Treatment in the spirit of this general calculation schema of the thermodynamics of the strongly correlated systems as Q2D cuprates in the normal state provides new insights in the meaning of the effective mass and other physical parameters of these systems.

Among the most amazing condensed matter phenomena is the orbital magnetic phase separation in clean normal metals at low temperatures, the well studied *Condon domains*. This instability is driven by the *magnetic interactions*, i.e. by exchange of photons between the conduction electrons.

We explain here, in a simple manner, how the sharp oscillations of the chemical potential should drive the quasi-two dimensional (Q2D)

dense electron systems to an orbital magnetic phase separation. These Ideally Conducting Phases (ICP) are the 2D limit of the 3D *Condon domains*. The unusual properties of the ICP's, namely the quantum Hall effect, expected to appear, even in absence of impurities, in each magnetic domain, attracts the theoretical and experimental attention for the last 20 years. Here we review shortly the main physical aspects of this phenomenon.

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1   Introduction

Magnetic quantum oscillations in metals [1] and semiconductors reflect the discrete part of the spectrum of electrons in high magnetic fields (the Landau levels [2]). Peierls has predicted in 1933 that the magnetic field dependence of the magnetic moment of an ideal two-dimensional electron gas (2DEG) should have a sharp saw-tooth form [3]. At that time 2D electron systems have not yet been fabricated and the Peierls theory was just an exercise in quantum statistics, illustrating the influence of the Landau quantization on thermodynamics of electron gas in magnetic field.

The discovery of the quantum Hal effect (QHE) [4, 5], has stimulated, during the last two decades, enormous interest to the physics of two dimensional electron systems under strong magnetic field.

The review is organized as follows. After a short Introduction, Section 1, where we describe some of the main phenomena which occur in different parts of the electron density vs magnetic field diagram, we develop, in Section *2* a two-level approach to the calculation of the magnetic quantum oscillations of the chemical potential in a free electron model. We apply. in Section 3, the obtained analytical expressions for the temperature and magnetic field dependence of the chemical potential, $\mu(B, T)$, to calculation of the thermodynamical functions such as magnetization, susceptibility, etc. The treatment of the quasi-two dimensional nonlinear de Haas-van Alphen effect and some properties of the Ideally Conducting Phases (ICP) [8], the 2D counterpart of the 3D Condon domains (see [9] for a detailed review on 3D Condon Domains), are described in Section *4*, notably their quantum Hall like properties. The thermodynamics of strongly correlated systems and the influence of the lattice structure on Landau levels is presented in Section 5. We are not touching here the problem of the coexistence of superconducting order and Landau quantization. A detailed review on dHvA oscillations in a superconductors is presented in [10].
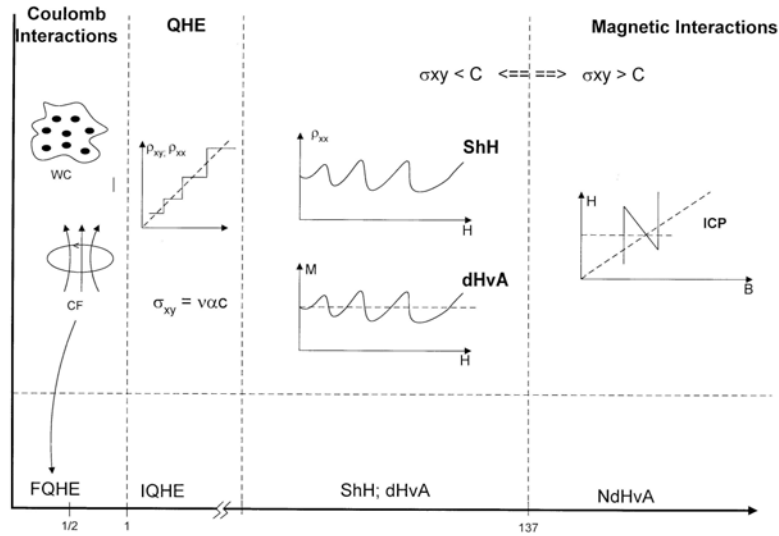
Figure 1: A concise diagram of most relevant physical phenomena in the two-dimensional electron systems under strong magnetic fields as a function as the filling factor, the horizontal axes.

Fig. 1 presents a concise diagram of most relevant physical phenomena in the two-dimensional electron systems under strong magnetic fields as a function the filling factor, the horizontal axes.

At low filling factors $\nu < 1$ the Coulomb interactions completely govern the physics of these systems and we have:

1. Wigner crystallization;

2. Fractional Quantum Hall effect;

3. Composite Fermions;

4. Integer quantum Hall effect (IQHE). At filling factors of the order of unity, the physics is governed by the sample properties, like the *localization* kind phenomena. This explains enormous precision of the celebrated quantum Hall effect plateaus.

At higher filling factors, the Shubnikov de Haas oscillations develop.

At rather high filling factors, i.e. at high electron concentrations, for a given magnetic field strength, which are typical for metals, the Coulomb interaction is effectively screened and the magnetic interactions, i.e. exchange

by the photons start to play a leading role in the thermodynamics of the electron system in a quantizing magnetic field. This results in the orbital magnetic phase separation in clean normal metals at low temperatures, the well studied *Condon domains.*

The Ideally Conducting Phases [8] are the 2D limit of the 3D Condon domains. The unusual properties of the ICP's are very similar to those in quantum Hall effect systems: zero resistivity within the domains and plateaus in Hall resistance. The difference is that the chemical potential is trapped in each magnetic domain within the Landau gap during finite intervals of magnetic fields due to diamagnetic quantum phase transition and not impurities, as is the case in semiconductors.

Let us remind first the main features of the celebrate quantum Hall effects: the integer one, IOHE, which originates in the pinning, by interplay between the localized and delocalized electron states, of the chemical potential in the middle of the Landau gap and the FQHE - the fractional quantum Hall effect, whose origin is in the electron correlations.

The most astonishing observation in IQHE is a set of precisely equidistant plateaus, during the magnetic filed sweep (or gate voltage) in the Hall resistance and concomitant zeros in the diagonal resistivity and conductivity components Fig. 2. The plateaus in the Hall coefficients reflect the fact that in QHE systems the chemical potential $\mu(H)$ is pinned to its value at zero field, the Fermi Energy $E_F \equiv \mu(H = 0)$, during the dHvA period. This is accounted for by localization-delocalization of 2D conduction electrons on Landau levels in presence of long range impurity potential.

In the absence of impurities the plateaus in Hall effect shrinks to points $\sigma_{xy} = \frac{nec}{H^*} = \nu\frac{e^2}{\hbar}$. Here $\nu = 1, 2, ...$ is an integer filling factor, n is the concentration of the delocalized electrons and H^* denotes the "center of plateau" values of the applied magnetic field. Thermodynamically this corresponds to the so called Peierls limit, where saw tooth magnetization oscillations should be expected [3]. As it was analytically shown in [8], in this, clean, limit the chemical potential is pinned to a Landau level during almost a whole dHvA period even at finite temperatures. It crosses the Landau gap at integer filling factors.

A renewed interest, in 80th, to the thermodynamics of the two-dimensional electron systems under strong magnetic fields was stimulated by the theoretical prediction [8] that a thermodynamic instability can pin the chemical potential within the Landau energy gap for the most of the dHvA period, resulting in the quantum Hall effect even in a clean 2DEG. Detailed study of the quantum statistics and thermodynamics of two dimensional electron systems at finite temperatures has been further developed in publications

by different groups [11-32]. Experimental studies were reported in: GIC's: intercalated graphite compounds [33]; heterojunctions [34-37], and in highly anisotropic organic materials [38-43].
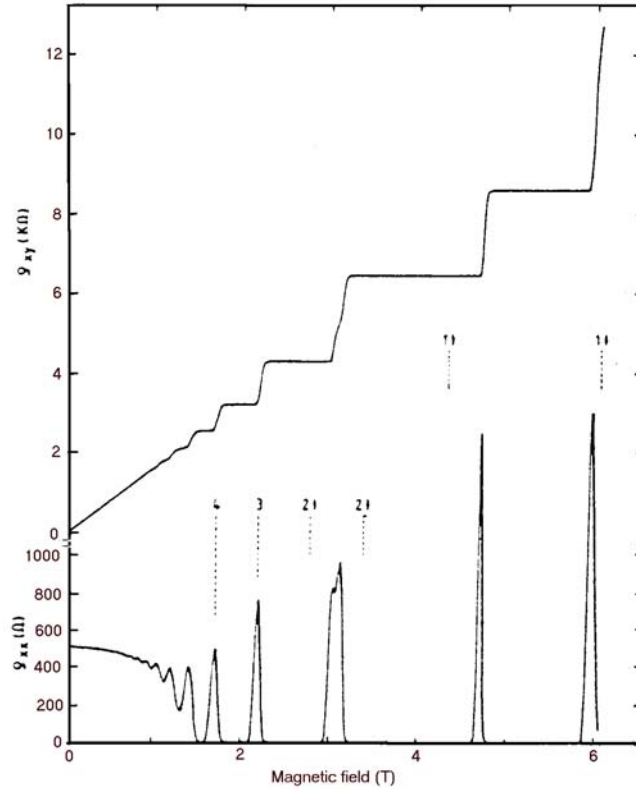

Figure 2: The quantum Hall effect: precisely equidistant plateaus in the magnetic field (or gate voltage) dependence of the nondiagonal resistivity (Hall effect) and concomitant zeros in the diagonal resistivity and conductivity tenzor components.

2 Oscillations of the chemical potential

2.1 Fermi energy and chemical potential

As it was outlined above already, the cornerstone for the understanding of statistics and thermodynamics of the two-dimensional electron gases in quantizing magnetic fields is the knowledge of the temperature and magnetic

field dependence of the chemical potential $\mu(B)$ [8, 11-32]. Huge degeneracy of the energy levels and large energy gaps between them, results in sharp magnetic oscillations of the chemical potential in these systems.
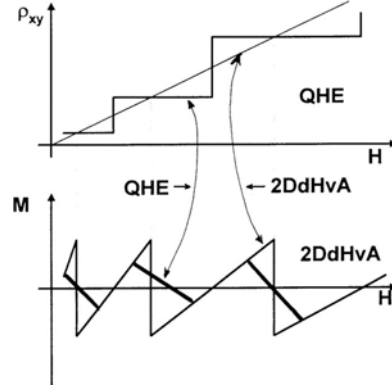

Figure 3: 2D dHvA as an anti-quantum hall effect: well developed plateaus in Hall conductivity appear when the saw-tooth magnetization oscillations are smeared.

The difference between the notions of the *Fermi energy* E_F and of the *chemical potential* $\mu(, T, B)$ can be understood as follows. According the basic definitions of the quantum statistics [44] the Fermi energy E_F is a normalization constant defined by the density of states of a system under consideration and the actual electron density. In 2DEG, where the density of the electronic states is energy independent, $E_F \sim n_o$, here n_o is the density of electrons.

The Fermi energy, E_F , now, is *the maximal occupied energy level in an electronic system at zero temperature and in the absence of external fields and at zero temperature.*

Using the definition of the *chemical potential* as a *minimal energy needed to add a particle to a statistical system* it is straightforward to see that at $T = 0$ and in the absence of external fields, the chemical potential coincides with the Fermi energy

$$\mu(B = 0; T = 0) \equiv E_F \qquad (1)$$

It is useful, therefore, to calculate the magnetic field dependence of the chemical potential, keeping E_F as a *a point of reference.*

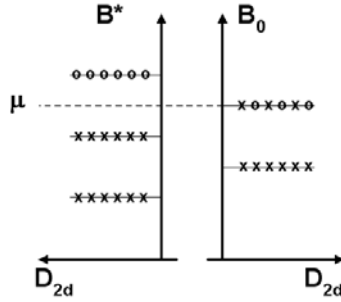Figure 4: B* and B_o are defined as the values of the magnetic field corresponding to the integer and half-integer filling factors respectively.

One should distinguish between three possible situations:

(a) The total number of electrons on the LL's is field independent, while the chemical potential is oscillating with an amplitude equal to the distance between the adjacent LL's. (N=const(B); $\mu = \mu(B)$). In this limit as we will later show in details, the *chemical potential* in *2DES* is "trapped" by a partially occupied Landau level during almost a whole dHvA period and will cross the Landau gap in infinitesimally narrow region around H^*, illustrated in Fig. 4, defined by the Eq. 3. This is visualized by the "Landau fan", Fig. 5, and in more details on Fig. 6.

(b) The chemical potential is field independent, while the number of electrons on the LL's is oscillating with field. (μ=const(B); N=N(B)).

(c) Both, the chemical potential and the number of electrons on the LL's are allowed to vary corresponding to the external conditions.

In what follows we assume that the total number of electrons on the LL's is field independent and start with the calculation of the magnetic quantum oscillations of the chemical potential.
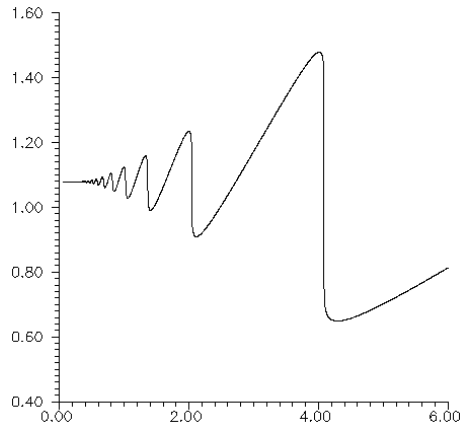
Figure 5: The Landau Fan: the magnetic field dependence of the chemical potential in 2DEG under strong magnetic field.
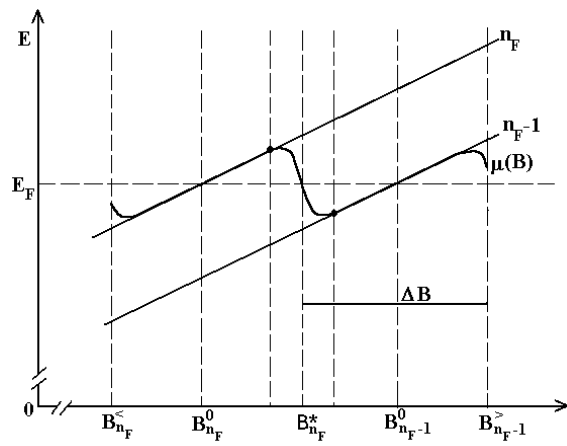

Figure 6: In a 2D electron gas, the Landau levels just below and just above the Fermi energy dominate the magnetic field dependence of the chemical potential.

2.2 Sharp Landau levels

Consider sharp (δ - functions) Landau levels at finite temperature. Equation, governing the magnetic field dependence of the chemical potential, at T $\simeq$ 0 reads:

$$g(B)\Sigma_{n=0}^{\infty} f_n = N \tag{2}$$

where $f_n = \left[1 + \exp\left(\frac{\hbar\omega_c(n+\frac{1}{2}) - \mu(B)}{k_B T}\right)\right]^{-1}$ is the Fermi distribution, $g(B) \equiv \frac{BS}{\Phi_o}$ is the degeneracy of the Landau levels, S -is the sample area and N is the total number of particles.

We present here the analytical calculation of the temperature and the magnetic field dependence of the chemical potential, in the framework of the two-level approximation, developed, originally, in [8]. Let n_F be the highest partially occupied (less than to a half of its degeneracy) Landau level, so that: $\hbar\omega_c(n_F\text{-} 1/2) < \mu < \hbar\omega_c(n_F+1/2)$. At typical experimental conditions: T $\simeq$ 1 K and H $\simeq$ 10T, $\alpha \equiv \frac{\hbar\omega_c}{2k_B T} \simeq 80$, provided that the effective mass is $m^* \simeq 0.1\ m_o$. Therefore all the terms in the sum of Eq. (2) up to $n = n_F\text{-}$ 2 could be replaced by $f_{n<n_F-2} = 1$ with an exponential: $e^{-\alpha} \ll 1$ accuracy, and the field dependence of the chemical potential is defined by the distribution of electrons on the levels $n_F\text{-}$ 1 and n_F. Eq. (2) can be cast in the form, expressing the filling of two uppermost LL's: $[1 + \exp[x_{nF} - \alpha]]^{-1} + [1 + \exp[x_{nF} + \alpha]]^{-1} = 1 + \tilde{n}$, where:$x_n \equiv \frac{n\hbar\omega_c - \mu}{k_B T}$ and $\tilde{n} \equiv \frac{n_s}{g} - n_F$.

Obtained equation can be reduced to an algebraic quadratic equation for y$\equiv e^{x_{nF}}$ which can be easily solved to give the following field and temperature dependence of the chemical potential in 2dEG+H [8]:

$$\mu\left(B,T\right) = \hbar\omega_c n_F - k_B T \mathtt{ln}\left[\frac{-\tilde{n}\mathtt{cosh}\alpha + \sqrt{1 - \tilde{n}^2\mathtt{sinh}^2\alpha}}{1 + \tilde{n}}\right] \tag{3}$$

Here $\tilde{n} = N_0 - g(n_F - 1)$ is the amount of electrons on the highest occupied Landau level.

Eq. (3) shows that at $T=0$: $\mu = \hbar\omega_c(n_F + 1/2)$ for all the values of the magnetic induction, apart of a discrete set $B = B^*$, where the "ln" term diverges. In a narrow region, around the value B* where $|A|\ exp^{\alpha} < 1$ the chemical potential is, approximately:

$$\mu \simeq \mu^* - k_B T A \frac{e^{\alpha}}{2} \tag{4}$$

where $\mu^* \equiv \hbar\omega_c^* n_F$. Therefore, at $B{\neq}B^*$the chemical potential is almost

pinned to a Landau level, and is approximately in the middle of the energy gap between the n_F-th and $n_F + 1$ LL's at $B \simeq B^*$, Fig. 6.

To show qualitatively that the derivative of the chemical potential with respect to B, in the vicinity of B*, is exponentially $\frac{\partial \mu}{\partial t} \simeq e^\alpha$ large , Eq. (4), let us consider, following [12], a two-level system, consisting of two adjacent Landau levels, with the chemical potential in between. Around B = B* the number of electrons N$_e^*$ in the upper LL and the number of "holes" in the lower LL are exponentially small: $N_e^* = N_h^* \simeq g^* \texttt{exp}^{-\alpha}$. When the magnetic induction $B = B^*$ is reduced by a small amount δB, the degeneracy of each LL is reduced by the amount $\delta g \simeq g^* \frac{\delta B}{B^*}$ so that δg electrons have to leave the lower level and to "jump" on the upper one. Since the lower level was already full, with an exponential: exp(-α) accuracy, its statistical weight will not change noticeable. The upper level, however, changes its occupation from exponentially small value: g^* $e^{-\alpha}$ to $\delta g \simeq g^* \frac{\delta B}{B^*}$.
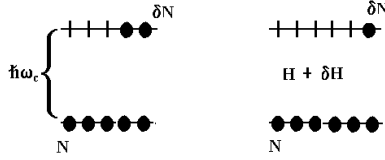

Figure 7: Increasing the magnetic field by an integer number of magnetic fluxes Φ_o results in increased degeneracy of the Landau levels, thus changing their statistical weight with respect to the chemical potential positioned in the Landau gap.

The location of the chemical potential is determined by the ratio between the number of electrons in the upper LL and the number of "holes" in the lower level: $\texttt{exp}[\frac{\delta \mu}{k_B T}]|_{\mu^*} \simeq \frac{N_e}{N_h} = 1 + \frac{\delta g}{N}$. For infinitesimally small δg one gets, therefore: $(\frac{\delta \mu}{\delta B})_{B^*} \simeq - k_B T \frac{g^*}{B^* N_h} \simeq - k_B T \frac{n_o \Phi_o}{2 B^{*2}} \texttt{exp}^\alpha$.

It is easy to show, now, that at T $\simeq 0$ the chemical potential is pinned to a Landau level for all but a discrete set B=B* of values of the magnetic induction. Assuming $| A | e^\alpha \gg 1$, we expand the Eq. 3 to obtain: $\mu \simeq \hbar \omega_c (n_F - 1/2) - k_B T \texttt{ln} \frac{(b_o + b)}{b_o - b})$ where: $b \equiv B - B_o$ is introduced.
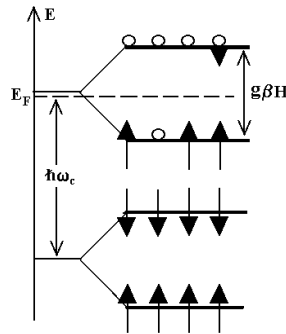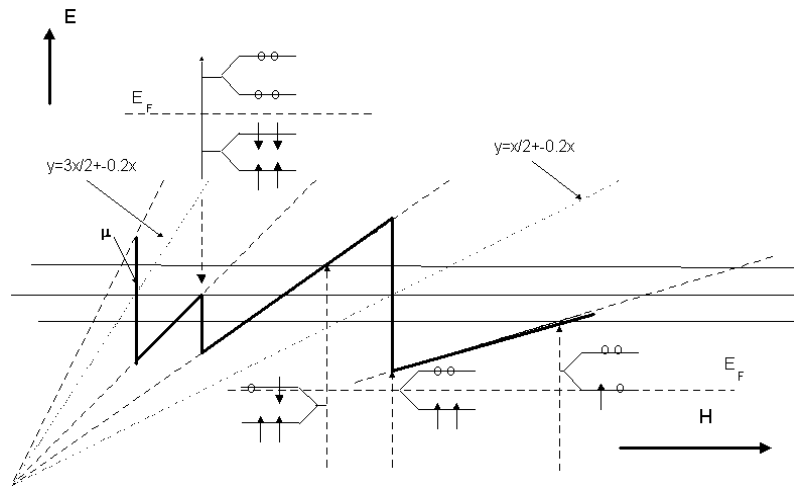
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Figure 8: Spin splitted 2DEG Landau levels.

Figure 9: Magnetic field dependence of the spin splitted 2DEG Landau levels.

2.3 Inclusion of spin

Let us assume, as usual, that the chemical potential is in the n_F-th, Zeeman spitted, Landau level, The population of the spin up and spin down branches of the n_F-th Landau level equals to the difference between the total number of electrons and those on the lowest, $n < n_F$, levels. Solving similar to the previous example we obtain the magnetic field dependence of the chemical potential, $\mu(B)$ to be:

$$\widetilde{\mu}(B) = -\ln[-\left(1-f_F^{-1}\right)chS + \sqrt{(1-f_F^{-1})^2ch^2S - (1-2f_F^{-1})}] \qquad (5)$$

here $\widetilde{\mu}(B) \equiv \frac{\mu(B)-\hbar\omega_c(n+1/2)}{k_BT}$, $f_{\sigma,n} = \frac{1}{1+\exp\left\{\frac{\hbar\omega_c(n+1/2)+\sigma\beta gH-\mu}{T}\right\}}$ and $s \equiv \frac{g\beta H}{T}$

This result schematically is presented in Fig. 9. Detailed experimental and theoretical study of the spin splitting resolved 2D dHvA in organic material is presented in [39].

2.4   Finite DOS between the Landau levels

The presence of localized electronic states within the magnetic energy gap such as, for example, impurity states or edge states, may lead to a significant slowing down in the variation of the chemical potential with respect to B, or even to the pinning of μ (a QHE situation). Following [12] we present here a qualitative interpretation for slowing down of the chemical potential by finite DOS in the Landau gap. Let us assume a constant DOS, D_{loc}, between LL-s. The number of electrons, trapped in the middle of the magnetic gap is $\delta\mu$ D_{loc}. The number of electrons on the upper level is now : $N_e = N_e^* + \delta g - D_{loc}^*\delta\mu$ and $\exp\frac{\delta\mu}{k_BT} = \frac{N_e}{N_h^*} = 1 + \frac{\delta g - D_{loc}^*\delta\mu}{N_h^*}$ or

$$\frac{\partial\mu}{\partial B}|_B^* \simeq \left[1 + \frac{D_{loc}}{N_{2d}}\frac{e^\alpha}{\alpha}\right]^{-1}\frac{\partial\mu}{\partial B}|_{B^*}^{free} \qquad (6)$$

where D_{2d} is the density of states of the 2d electron gas. We see that exponentially small density of states (DOS) between the Landau levels is sufficient to slow down the motion of the chemical potential through the magnetic gap.
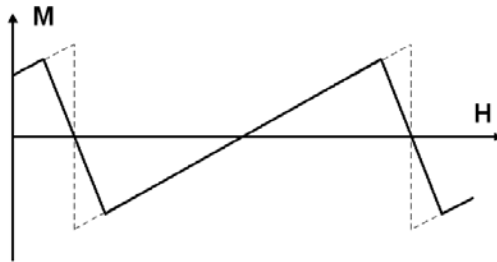

Figure 10: Magnetization oscillations for the 2DEG with a finite DOS of electron states in the Landau energy gap.

The condition under which $\frac{\partial \mu}{\partial B}$ retains its intrinsic behavior (i.e. $\propto \frac{e^\alpha}{\alpha}$) is that the density of localized states within the magnetic energy gap should be exponentially small with respect to the density of states of the 2D free electron gas: $D_{loc}(E) < 4\alpha e^{-\alpha} D_{2d}^{free}$

2.5 Landau level broadening

The broadening of the Landau levels has different origins and the shape of the LL's has different forms, respectively. The most important broadening mechanisms are the scattering by the short range impurities (Gaussian form, usually), long range impurities (Lorentzian form in most cases) etc. Depending on the sample and external parameters the broadening by the same source may have different forms in different limits of parameters, as temperature, magnetic field, sample structure etc. .We will consider here just one example of broadening, namely of a form:

$$D_o \int_{-\infty}^{\infty} \frac{dx}{\text{ch}(ax)} = g(H) \tag{7}$$

where D_0 is the density of states in the middle of a Landau level and Γ is its width. Given Γ we can find D_0 from Eq. 7: $2\frac{D_o}{a}\text{arctg}e^{ax}|_{-\infty}^{+\infty} = \frac{\pi D_o}{a} = g(H)$ to get: $D_o = g(H)a/\pi$. The normalization condition for the chemical potential is: $D_o\int_{-\infty}^{\mu} \frac{dx}{\text{ch}(ax)} = A(B)$.
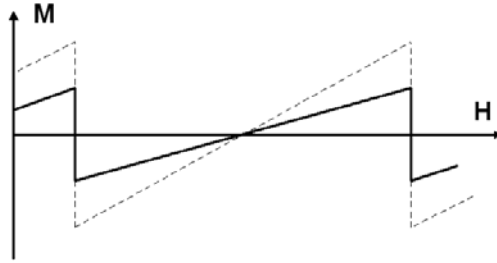

Figure 11: Magnetization oscillations of 2DEG with broadened Landau levels.

Performing this integral, we arrive at $e^{a\mu} = \text{tg}\pi/2A$ and obtain the following expression for the field dependence of the chemical potential: $\mu = \hbar\omega_c(n_F - .5) + \frac{1}{a}$ ln $\text{tg}(\frac{\pi}{a}A)$.

Two Landau levels In order to get a smooth transition of the chemical potential between the neighboring Landau levels, we shall calculate the magnetic field dependence of $\mu(B)$ in the case, when two adjacent Landau levels are partially occupied (i.e. we take into account the tail in the density of states of the n_F - 1 Landau level, while μ is in the n_F-th LL): $\int_{-\infty}^{\mu} \frac{dE}{\text{ch}(aE)} +$ $\int_{-\infty}^{\mu} \frac{dE}{\text{ch}[a(E-\hbar\omega_c)]}$ $= \frac{Ag(H)}{D_o} = g(H)/D_o$E $\Rightarrow \mu$;E- $\hbar\omega_c \Rightarrow \mu - \hbar\omega_c$.
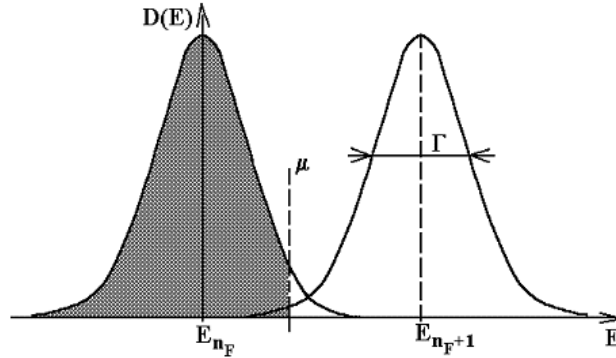

Figure 12: Lorentzian broadening of Landau levels in 2DEG.

Performing the integration we get: $\text{arctg}(e^{a\mu}) + \text{arctg}(e^{a\mu 2-\hbar\omega_c}) = A\pi/2$. Using the relation: $\text{arctg}x + \text{arctg}y = \frac{(x+y)}{(1-xy)} + \pi$ $(x > 0, xy > 0)$ we obtain:

$$\mu = \hbar\omega_c(n_F - .5) + \frac{1}{a}\ln[\frac{-(1+\gamma)\pm\sqrt{(1+\gamma)^2 + 4\gamma\text{tg}^2(A)}}{2\gamma\text{tg}(A)}]$$

where: $e^{a\mu} = x$ and $e^{-a\hbar\omega_c} = \gamma$ are introduced.

2.6 Finite k_z-dispersion

In order to account for a finite warping of the Fermi surface one should include in the calculation of the chemical potential a finite k_z electron dispersion [28, 29]:

$$N = \frac{ga}{2\pi\hbar} \sum_{n=0}^{\infty} \int_0^{\frac{2\pi\hbar}{a}} f_n dp_z \tag{8}$$

where g is the degeneracy of a LL, a is the lattice constant in z direction (the 2D layer separation in heterostructures) and N – is the number of particles

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in a single layer. The chemical potential is situated between the two LL's with energies $\hbar\omega_c(n_F - \frac{1}{2})$ and $\hbar\omega_c(n_F + \frac{1}{2})$. We assume $\hbar\omega_c \gg kT$ so that only these two Landau levels make contribution to thermodynamics. One can set $f_n = 1$ for all $n < n_F - 1$ and $f_n = 0$ for all $n > n_F$:

$$N = (n_F - 1)g + \frac{ga}{2\pi\hbar} \int_0^{\frac{2\pi\hbar}{a}} (f_{n_F - 1} + f_{n_F})dp_z \ . \tag{9}$$

The energy of electron is given by $E = \hbar\omega_c(n + \frac{1}{2}) + E_z$, where n is the number of LL and the term $E_z = \frac{W}{2}(1 - \cos(\frac{p_z a}{\hbar}))$ gives the dependence of energy on the momentum along z axis. For ideal two-dimensional case $W = 0$, and $W \neq 0$ describes the warping of Fermi surface. Later for convenience we shall use another expression for E_z :

$$E_z = \frac{W}{2} \cos\left(\frac{p_z a}{\hbar}\right). \tag{10}$$

The difference between expression (10) and the previous one consists of only shifting the starting point of energy by a constant $\frac{W}{2}$ (which of course does not change any physical result) and the shifting of starting point of quasi-momentum p_z by $\frac{\pi\hbar}{a}$ (that makes no difference because of subsequent integration over the full period of p_z).

In order to calculate the integral in (9) we use the condition

$$\exp\left(-\alpha + \frac{W}{2kT} \pm X_F\right) \ll 1 \tag{11}$$

which even for $W \sim \hbar\omega$ is equivalent to $\sqrt{\alpha} \gg 1$ and is usually fulfilled in the experiments. Expanding over small parameter (11) and keeping only first two terms one arrives at the following expression for the chemical potential

$$\mu = \hbar\omega_c n_F + kT\,\text{arch}\left(\frac{\tilde{n}e^{\alpha}}{2I_0(\frac{W}{2kT})}\right). \tag{12}$$

Here the notation $\tilde{n} = \frac{N}{g} - n_F$ is used and $I_0(\frac{W}{2kT})$ is the modified Bessel function of argument $\frac{W}{2kT}$.

It is easy to see that in the limiting case $\frac{W}{2kT} \ll 1$, formula (12) coincides with the expression for chemical potential without warping of Fermi surface, Eq.(3), obtained in [8].

3 Thermodynamic potentials

3.1 Thermodynamic relations in magnetic fields

Here we remind the definitions of different thermodynamic potentials, [1, 44] which will be used in following sections:

The Helmholtz Free energy is :

$$F = U - TS \tag{13}$$

where:

$$dU = TdS + \mu dN - MdB \tag{14}$$

is the internal energy, T is the temperature, S is the entropy of a system and μ is the chemical potential. The differential relationship reads:

$$dF = \mu dN - MdB - \delta dT \Rightarrow M = -\left(\frac{\partial F}{\partial B}\right)_{V,T,N} \tag{15}$$

The Gibbs thermodynamic potential Ω is connected with the Helmholz Free energy F by the following relation:

$$F = \Omega - N\mu \tag{16}$$

and in differential form:

$$d\Omega = -MdB - \delta dT - Nd\mu \Rightarrow M = -\left(\frac{\partial \Omega}{\partial B}\right)_{V,T,\mu} \tag{17}$$

In what follows we will be interested in the derivatives of these potentials with respect to either H or B.

Magnetization is defined, therefore, either:

$$M(B) = -\frac{dF(B)}{B}|_{N=\texttt{const}} \tag{18}$$

or

$$M(H) = -\frac{d\Omega(H)}{H}|_{\mu=const} \tag{19}$$

depending on wether the electron density or the chemical potential is fixed.

In the next section we develop a simple procedure, based on the previously described two-level approximation, for calculating the magnetic field and temperature dependence of the thermodynamic potentials and its derivatives in 2DEG in quantizing magnetic field.

3.2 Calculation of free energy

Magnetization is defined as a derivative with respect to magnetic field of either the Free energy, F, Eq. (18) (constant number of particles) or of the thermodynamic potential Ω, Eq. (19) (constant chemical potential).

Let us calculate the analytical expression for the magnetic field dependence of the free energy $F(B) = E(B) - TS$ in the simplest case of sharp (δ - functions) Landau levels. The thermodynamical potential $\Omega(B)$ can be written in the form:

$$\Omega(B) = k_B T g(B) \sum_{n=0}^{\infty} \ln(1 - f_n) \qquad (20)$$

We employ here the approximation, similar to the one, used previously for calculation of the magnetic field and the temperature dependence of the chemical potential, and arrive at the following expression for the thermodynamic potential:

$$\Omega(B) \simeq -g(B)[-\mu(B)n_F - \hbar\omega_c \frac{n_F^2 + 1}{2} + k_B T \ln\left(2\cosh x_F + 2\cosh\alpha\right)] \quad (21)$$

where: $x_n = \frac{n\hbar\omega_c - \mu}{k_B T}$.

The magnetic field dependence of the Free energy of a two- dimensional electron gas in a magnetic field is given in Fig. 13.
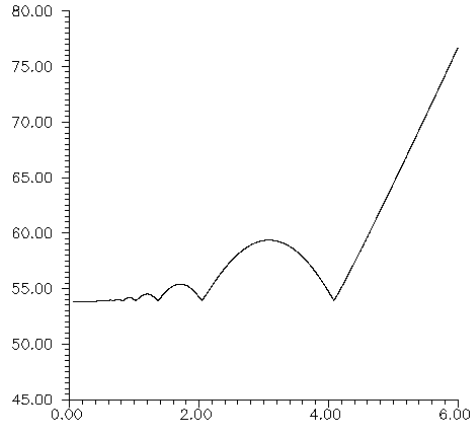

Figure 13: The magnetic field dependence of the Free energy of a 2DEG in a magnetic field.

For finite k_z dispersion the following expression for the thermodynamical potential has been calculated [28, 29].

$$\Omega = -kT \frac{ga}{2\pi\hbar} \int_0^{\frac{2\pi\hbar}{a}} \sum_{n=0}^{\infty} \ln\left(1 + e^{\frac{\mu - E_n(p_z)}{kT}}\right) \, dp_z \tag{22}$$

The calculation of sum and integral is analogues to those for chemical potential. In the approximation $W \ll \hbar\omega_c$ the free energy is calculated to be:

$$F = \Omega + \mu N == g\left\{\frac{n_F{}^2}{2}\hbar\omega_c + \left(\frac{N}{g} - n_F\right)\mu - kT \, e^{-\alpha} \cdot 2\cosh X_F \cdot I_0\left(\frac{W}{2kT}\right)\right\} \tag{23}$$

3.3 Magnetization

Differentiating the Free energy, with respect to magnetic field and using the obtained expression for $\Omega(B)$, we obtain an analytical expression for the magnetization of 2DEG at finite temperatures, [8]:

$$\begin{aligned}
M(B) &= -\frac{\partial F(B)}{\partial B} = \frac{2}{\phi_0}[n_F\mu(B) + \hbar\omega_c\left[n_F + \frac{\sinh\alpha}{2\sinh(x_F)}\right] - \\
&\quad \hbar\omega_c(n_F^2 + 1) + k_B T\ln[2\cosh(x_F) + 2\cosh\alpha]]
\end{aligned} \tag{24}$$

where $A \equiv n_F - \frac{n_s}{g}$.

In order to guide an experimental study of dHvA in 2DEG, one should obtain an analytical expression for the envelope of the magnetization oscillations, i.e. the 2D analog of the LK formula.

The maxima (minima) of the magnetization, could be obtained from the Eq. (24) by equating to zero the derivative $\frac{\partial M}{\partial B_{extr}} = 0$. It follows, that:

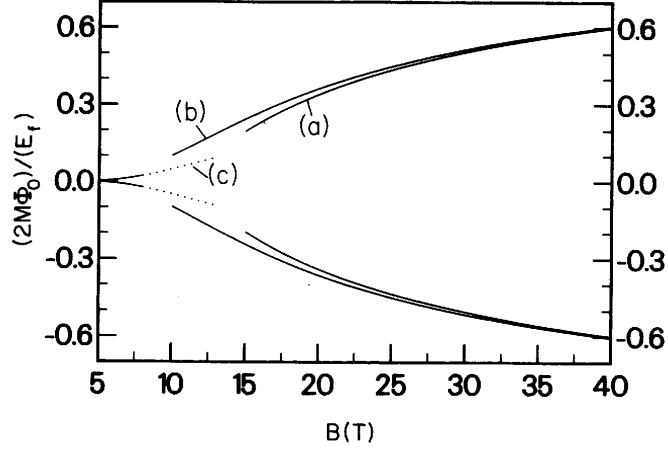$$B_{extr} = B^*\left[1 \mp \frac{1}{2\alpha^*(n_F \pm 1)}\right] \tag{25}$$

Figure 14: Envelop of the magnetization oscillations in 2DEG.

Keeping in mind that $\frac{1}{2\alpha^*(n_F \pm 1)} \ll 1$ we substitute B_{extr}, defined by Eq. (25), into Eq. (24) and obtain the analytical behavior of the magnetization amplitude [19]:

$$M_{extr} \simeq \pm \frac{E_F}{\phi_o} \left[1 - \frac{1}{\alpha^*} \ln\left(2\alpha^*\right) - \frac{1}{\alpha^*} \right] \qquad (26)$$

Inclusion of the finite k_z dispersion yields the following expression for the magnetization [28, 29]:

$$
\begin{aligned}
M \quad = \quad & \frac{S}{\Phi_0}\hbar\omega_c\Big\{ -n_F\tilde{n} + \frac{n_F}{2\alpha}\mathrm{arsh}\left(\frac{\tilde{n}e^\alpha}{2I_0(\frac{W}{2kT})}\right) + \\[2mm]
& + e^{-\alpha} \cdot I_0\left(\frac{W}{2kT}\right) \cdot \cosh X_F \cdot \left(-1 + \frac{1}{\alpha}\right) \Big\}
\end{aligned}
\qquad (27)
$$

This is the expression for magnetization with warped Fermi surface. The only restriction on the magnitude W of the warping is that it must be not greater than energy difference between LL's i.e. .

The values B_{ex} of magnetic field at which magnetization M has extrema are given by

$$B_{ex} = B^* \left(1 \pm \frac{1}{2\alpha(n_F + 3/2)} \right) \qquad (28)$$

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and the envelope of magnetization is obtained to be:

$$M_{\pm} = \pm \frac{SE_F}{2\Phi_0} \left\{ \frac{1}{\alpha} \text{arsh} \left(\frac{e^\alpha}{4\alpha I_0(\frac{W}{2kT})} \frac{n_F}{n_F + 3/2} \right) - \frac{1}{\alpha} \frac{n_F + 1}{n_F + 3/2} \right\} \quad (29)$$

In the limit $\frac{W}{2kT} \ll 1$ and $n_F \gg 1$ this formula coincides with one, obtained for the ideal two-dimensional case [19] : $M_{\pm} = \pm \frac{SE_F}{\Phi_0} \left[1 - \frac{1}{\alpha}\ln(2\alpha) - \frac{1}{\alpha} \right]$.

In the inverse case of large warping formula (29) becomes:

$$M_{\pm} = \pm \frac{SE_F}{\Phi_0} \left[1 - \frac{W}{\hbar\omega_c} + \frac{1}{\alpha}\ln\left(\frac{\sqrt{kT \cdot \pi W}}{\hbar\omega_c} \right) - \frac{1}{\alpha} \right], \quad (30)$$

This formula differs substantially from the ideally-2D one, Eq. (26). The amplitude of the oscillations depends approximately linearly on the size of warping W.

3.4   Susceptibility

As it was shown above, the saw tooth form of the magnetization of the two-dimensional electron gas in a magnetic field results in a constant, between the adjacent B* orbital susceptibility: $\chi_{orb}=$ dM(B)/dB.

At finite temperature the expression for the 2D magnetic orbital susceptibility is given by the derivative of the magnetization, Eq. (30). This expression defines χ in the case when the chemical potential lies in the n_F-s Landau level. Between the two adjacent Landau levels the susceptibility should take, at T=0, infinite negative values. Finite temperature smooths out this unphysical "jumps" and, as it was shown in [12], results in exponentially $(exp\frac{\hbar\omega_c}{k_BT})$ sharp spikes.

When the Landau levels are broadened, the susceptibility oscillations are smoothed out, as in Fig. 15. It is also shown in [12], that in 2D the differential relationship between M(B) and Ω turns to be an algebraic one, that is:

$$M(B) = -\frac{\Omega(B)}{B} - \frac{E(B)}{B} \quad (31)$$

Differentiating, now, M with respect to B, we obtain:

$$\chi(B) = -\left(\frac{\partial^2 F(B)}{\partial B_2} \right)_{n_s,T} = \frac{1}{B} \left[n_s \frac{\partial\mu}{\partial B} - \frac{\partial E}{\partial B} \right] \quad (32)$$

One can see, from this relationship, that χ_{orb} is proportional to the $\frac{\partial\mu}{\partial B}$ in a narrow vicinity of B*.
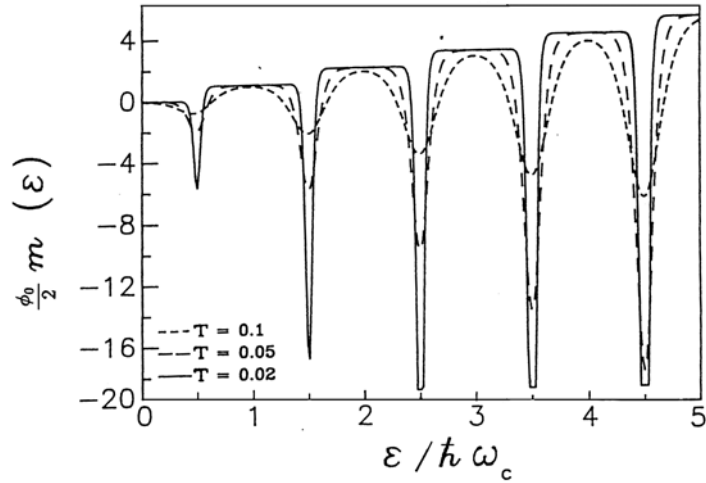
Figure 15: Susceptibility oscillations of 2DEG with broadened Landau levels.

Since E(B) $\propto B^2$ the second term is const(B), between the B^* s. The second term in Eq. (32) involves the derivative of E(B) with respect to B. Unlike the first term this one does not diverge in the zero temperature limit since E(B) is a continuous function of B at T=0. Therefore the dominant contribution to the spikes in the OMS originates in the chemical potential derivative with respect to B.

Within this narrow region the OMS is, therefore, a direct measure of the derivative of the chemical potential μ with respect to B, so that by measuring experimentally the magnetic susceptibility one can get an important information on the magnetic field dependence of the chemical potential in two-dimensional electron gas under strong magnetic fields.

4 Ideally conducting phases

4.1 Magnetic interactions

Among the most amazing condensed matter phenomena is the orbital magnetic phase separation in clean normal metals at low temperatures, the well studied *Condon domains*. This quantum phase transition is driven by the *magnetic interactions*, i.e. by exchange of photons between the conduction electrons [45]. As it was outlined, for the first time by D. Shoenberg [46], the energy spectrum of electrons in a metal under strong magnetic fields is

defined by magnetic induction $B = H + M(B)$, rather than by the external field H (the Shoenberg conjecture). This means that the dynamics of each charge is influenced by a magnetic moment produced by all the remaining charges [1, 46, 47]. As a result, a self-consistent nonlocal (current-current) interaction [45] is produced between the charges. The difference between H and $M(B)$ becomes significant when the amplitude of magnetization oscillations is comparable with their period. Under such conditions the maximal differential susceptibility $\chi_m = \frac{\partial M}{\partial B}$ approaches $1/4\pi$ and the electron system becomes unstable toward magnetic phase separation.

The microscopic theory starts with the Hamiltonian [45]:

$$H_{HNP} = H_0^\gamma + \frac{\hbar^2}{2m}\left(\overrightarrow{\nabla} + \frac{ie}{\hbar c}\overrightarrow{A}\right)\psi^+ \left(\overrightarrow{\nabla}\frac{ie}{\hbar c}\overrightarrow{A}\right)e\psi + U \qquad (33)$$

Here H_0^γ is the free radiation field part of the HNP Hamiltonian density, ψ is the second-quantied electron field and U contains all other, non-electromagnetic electron interactions.

The self-consistent vector-potential is defined by

$$\overrightarrow{A} = e\overrightarrow{A}_o + \overrightarrow{A^r} \qquad (34)$$

where

$$\overrightarrow{\nabla} \times \overrightarrow{A}_o = \overrightarrow{H_o} \qquad (35)$$

are the external fields and

$$\Box^2 \overrightarrow{A^r} = -\left[\overrightarrow{j} - \frac{1}{\nabla^2}\left(\overrightarrow{\nabla} \cdot \overrightarrow{j}\right)\right] \qquad (36)$$

is the radiation field, arising from currents induced in the medium, respectively. Here $\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ and the operator for the electric current is

$$\overrightarrow{j} = -\frac{\partial H_{HNP}}{\partial \overrightarrow{A}} = \frac{e\hbar}{2imc}\left(\psi^+\overrightarrow{\nabla}\psi - \overrightarrow{\nabla}\psi^+\psi\right) - \frac{e^2}{mc^2}\overrightarrow{A}\psi^+\psi \qquad (37)$$

4.2 Condon domains

In sufficiently pure three-dimensional metals, the magnetization may approach the value of the dHvA period. This will result in a phase transition to a new electronic state with inhomogeneous magnetization (Condon Domains). The phenomenon of domains associated with the orbital magnetic

moment has been discovered in different metals as: beryllium [48], silver [49] and aluminium [50].

Let us consider the stability of the electron gas in magnetic field using the Pippard's phase diagram in H-B plane, Fig. 16. In the high-field portion of the diagram, where the slope $\chi_n(n_F)$ of the M(B) curve is smaller than $\frac{1}{4\pi}$, the slope of the H(B) curve is positive and for each value of external field H in this region there is only one value of B on the H(B) curve.

If the field H is lower, such that

$$\chi(B) > \frac{1}{4\pi} \tag{38}$$

the slopes of the linear portions of the curve are always negative and for each value of H there are at least three different values of B on the H(B) curve.
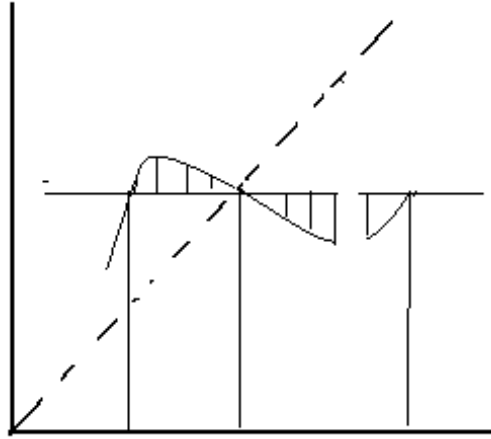
Figure 16: The Pippard constuction for the nonlinear de Haas van Alphen effect.

The coexisting Condon domains represent different phases of a metal with different magnetization. Thus, the transition from the homogeneous phase to the domain one can be considered as a diamagnetic phase transition occurring in each dHvA cycle when sweeping an external magnetic field at $4\pi\chi_m > 1$. At a fixed external field, a temperature decrease causes an increase in the magnetization amplitude, and, therefore, drives the diamagnetic phase transition.

The structure of the Condon domains in three-dimensional metals was studied theoretically in the 60th and early 70th: [45,54-57] and was generalized in 90th to include dynamic and kinetic phenomena, see for a recent review [9].

4.3 Ideally conducting phases

Renewed interest in these phase transitions for electron gases under strong magnetic fields was stimulated by the theoretical prediction, [8] that the sharp oscillations of the chemical potential in two-dimensional dense electron system may result in a domain state with the quantum Hall effect properties in each domain. Namely, within each domain the chemical potential is pinned between two adjacent Landau levels Fig. 18, i.e. to a particular quantum-Hall plateau, with Hall resistivity quantized and longitudinal resistivity vanishing.

Such a *nondissipative, QHE - like,* state, presents the *Ideally Conducting Phases* (ICP): a two-dimensional manifestation of Condon domains. In [20] the Ginsburg-Landau equation, corresponding to this phase transition, is developed.

The corresponding H-B diagram in the two-dimensional case is presented in Fig. 17

For all values of B between B_c^1 and $B_c^{II} \equiv \frac{B_c^1}{\sqrt{2}}$ there are no more than three values ($B_<$, B_0 and $B_>$) on the $H(B)$ curve for each value of H. Using the *Maxwell construction*, we observe that the stable solutions of the equation:

$$B = H + 4\pi M(B) \tag{39}$$

in the case of Q2DEG correspond to the Quantum Hall Effect-like nondissipative state.

Indeed, the negative slope on the H vs B curve appears at the field:

$$H^I = \left[\frac{2m_c \epsilon_F^2}{\hbar^2 d} \right]^{\frac{1}{2}} \tag{40}$$

which corresponds to the number of filled Landau levels, given by the following expression:

$$n_F = \sqrt{\frac{m^* c^2 d}{2e^2}} \tag{41}$$

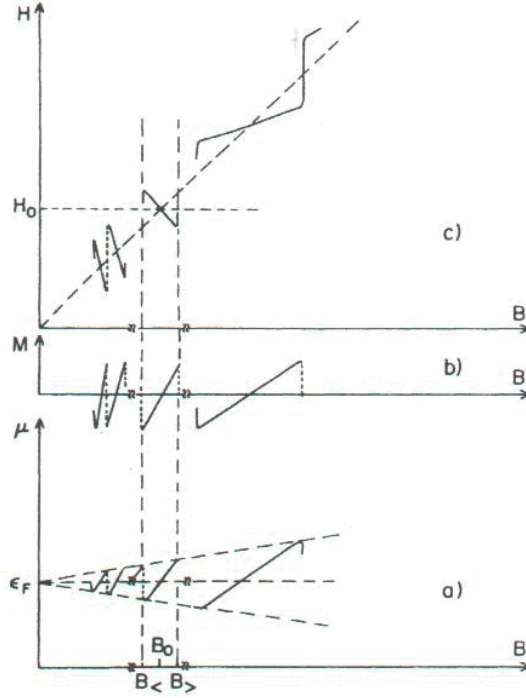where d stands for the distance between the electron layers in the case of a quasi-two dimensional layered conductor.

Figure 17: Schematic illustrations of (a) the chemical potential, (b) the magnetization M, and (c) the magnetic field H as functions of the magnetic induction B at zero temperature. $B_<$, $B_>$, and B_0 are defined in the text and ϵ_F is the value of the chemical potential at zero magnetic field

Therefore, applying the Maxwell construction to determine the equilibrium condition for coexistence between the two phases, one finds that the chemical potential for the entire system is located at

$$\mu = \hbar\omega_c^> n_F = \hbar\omega_c^< (n_F + 1) = n_s \frac{2\pi\hbar^2 d}{m_c} \qquad (42)$$

where $\omega_c^> \equiv \frac{eB_>}{cm^*_{xy}}$, $\omega_c^< \equiv \frac{eB_<}{cm^*_{xy}}$, $B_> \equiv B_{n_F} = \frac{(n_o\phi_0)}{(n_F)}$ and $B_< \equiv B_{n_F+1} = \frac{(n_o\phi_0)}{(n_F+1)}$ where n_o is the areal density of electrons, which is constant throughout the sample and $\phi_0 \equiv \frac{hc}{e}$ is the flux quantum. Thus *each phase plays the role of an electron reservoir for the other phase*, keeping the chemical potential fixed in a midgap position.

In other words, for each phase the chemical potential is located in the

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Landau gap, i.e. separates the highest occupied Landau level from the level just above it so that the ideal conductivity is expected to occur in the interior of each phase. Significant deviations from ideal conductivity may occur in the transition zones separating different phases but the overall effect on the conductivity of the whole sample should be small because of the small size of the transition zones.
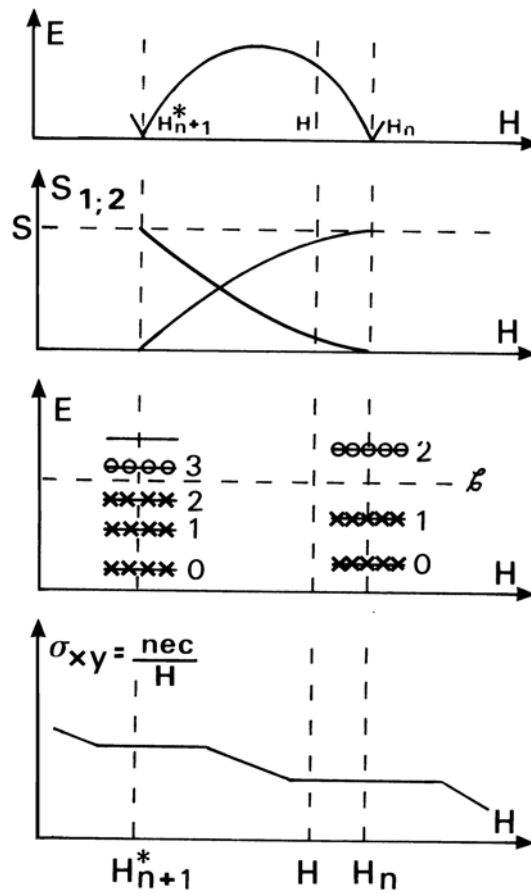

Figure 18: a) The magnetic field energy of the two-dimensional electron gas; b) The variation, with external magnetic field, of the areas $S_{1,2}$ of two different phases; c) The position of the chemical potential in neighboring ICP's; d) The plateuas in the Hall conductivity, expected in the Ideally Conducting Phases.

Below the critical field ($B_c = n_o \frac{\phi_0}{n_c}$) the translational symmetry in this system is broken and the self-consistent field B becomes spatially inhomogeneous. If the field varies slowly over a cyclotron radius: $R_c = a_H \sqrt{(2n_F)}$, $a_H \equiv \sqrt{\left(\frac{c\hbar}{eB}\right)}$ is the magnetic length, one may define local Landau levels by using an average of the inhomogeneous field over a cyclotron orbit.

The magnetic field value, corresponding to the first appearance of more than three solutions for Eq. (39), as can be easily seen from the Fig. 19 is defined by the condition:

$$H\left(B_{n_F}^<\right) = H\left(B_{n_F-1}^>\right) \tag{43}$$

This condition yields:

$$H^{II} = \frac{1}{\sqrt{2}} H^I \tag{44}$$
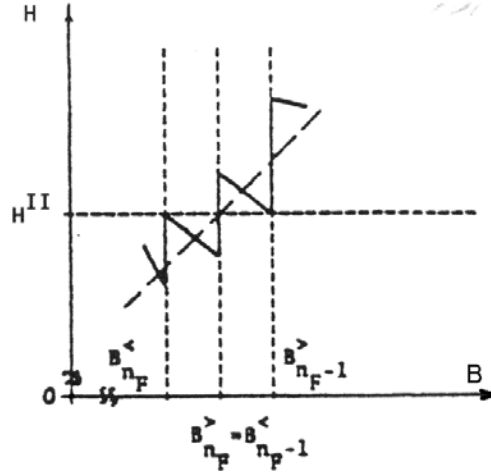
Figure 19: The minimal value of the magnetic field, corresponding to a threefold solution of the Pipparrd construction..

Since B(x) varies between the values $B_< \equiv \frac{(n_o\phi_0)}{(n_F+1)}$ and $B_> \equiv \frac{(n_o\phi_0)}{n_F}$ of the field deep inside each domain, one may define a mean field for the domain wall region by:

$$B_0 \equiv \frac{1}{2}(B_< + B_>) \simeq B_{n_F} \tag{45}$$

and a fluctuating field:

$$B_1(x) \equiv B(x) - B_0 \tag{46}$$

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with $|B_1(x)| < \frac{B_0}{2n_F} << B_o.$.

The coefficients of the Ginzburg - Landau equation for the diamagnetic phase transition into a two-dimensional Condon domain state have been calculated microscopically in [20] and explicit expressions for the critical temperature and for the temperature dependent coherence length have been obtained. The obtained analytical expressions permits to calculate the effect of critical fluctuations on the diamagnetic susceptibility above the critical temperature.

The coefficients of this expansion can be directly calculated from the closed analytical expression for M(B) obtained in Ref.[8]. The expressions for the susceptibility $\chi(B_0)$ and its first and second derivatives are thus [20]:

$$\chi(B_0) = \chi_0 - \chi_0 \frac{(4k_BT)}{\hbar\omega_{c,0}} \tag{47}$$

$$\left(\frac{\partial\chi}{\partial B}\right)_{B=B_0} = 0 \tag{48}$$

$$\left(\frac{\partial^2\chi}{\partial B^2}\right)_{B=B_0} = -\frac{\chi_0}{b_0^2}\frac{(8k_BT)}{\hbar\omega_{c,0}} \tag{49}$$

and

$$b_0 \equiv \frac{B_0}{2n_F}. \tag{50}$$

Thus the resulting equation for $b_1(x) \equiv \frac{B_1(x)}{b_0}$ has a Ginzburg-Landau like form:

$$\epsilon b_1(x) - \xi_0^2 \frac{d^2}{dx^2}b_1(x) + b_1^3(x) = 0 \tag{51}$$

where the coherence length ξ_0 is given by:

$$\xi_0^2 \equiv \left(\frac{3}{32}\right) r_c^2 \frac{(k_BT)}{\hbar\omega_{c,0}} \tag{52}$$

and $\epsilon \equiv \frac{3}{2}\left(\frac{(T-T_c)}{T}\right)$. The critical temperature T_c can be readily obtained from the requirement: $1/4\pi - \chi(B_0) = 0$ with $\chi(B_0)$ defined above. The result is:

$$T_c = \frac{\hbar\omega_{c,0}}{4k_B}\left(\frac{1-n_c^2}{n_F^2}\right) \tag{53}$$

where n_c is the critical value of n_F, or in terms of the critical field B_c by $n_c = \frac{(n_o\phi_0)}{B_{c.}}$.

The non - uniform solutions of Eq. (51) below $T_c are$:

$$b_1 (x)=\pm\sqrt{|\epsilon|}tgh \left[\frac{x}{\xi (T)}\right] \tag{54}$$

where $\xi (T)$ is the temperature dependent coherence length given by:

$$\xi (T)\equiv\frac{(\sqrt{2}\xi_0)}{\sqrt{|\epsilon|}} \tag{55}$$

Eq. (51) is analogous to the G.L. equation describing a domain wall between the superconducting and the normal regions in the intermediate state of a type-I superconductor

5 Magnetization currents

Let us start with

$$b = h + 4\pi \left[M (B_o + b) - M (B_o)\right] \tag{56}$$

where h is the magnetic field, b is the magnetic induction of the wave and M is the magnetization, defined by

$$M = -m \sum_{r=1} A_r \sin \left[2\pi r \left(\frac{F}{B}\right) + \varphi_r\right] \tag{57}$$

which is the Landau-Lifshits-Kosevich formula, with

$$F = \frac{heS_{extr}}{2\pi c} \tag{58}$$

5.1 Nonlinear dynamics in ideally conducting phases

Due to the nondissipative nature of the ICP's a rich dynamics of the fields and currents should be expected. Indeed as we can see from Fig. 20 the logistic map constructiion of the Feigenbaum kind theory of chaotic phenomena can be applied to study the unusual electrodynamics of two-dimensional electron systems under strong magnetic fields. This is in fact a nonlinear and a nonlocal Chern-Simon electrodynamics, studied in details in [64-67] the dHvA frequency.

For b small compared to the dHvA period, we can use linear appoximation

$$h = b - 4\pi\frac{\partial M}{\partial B}\lfloor_{b=0}b = \mu_e b \tag{59}$$
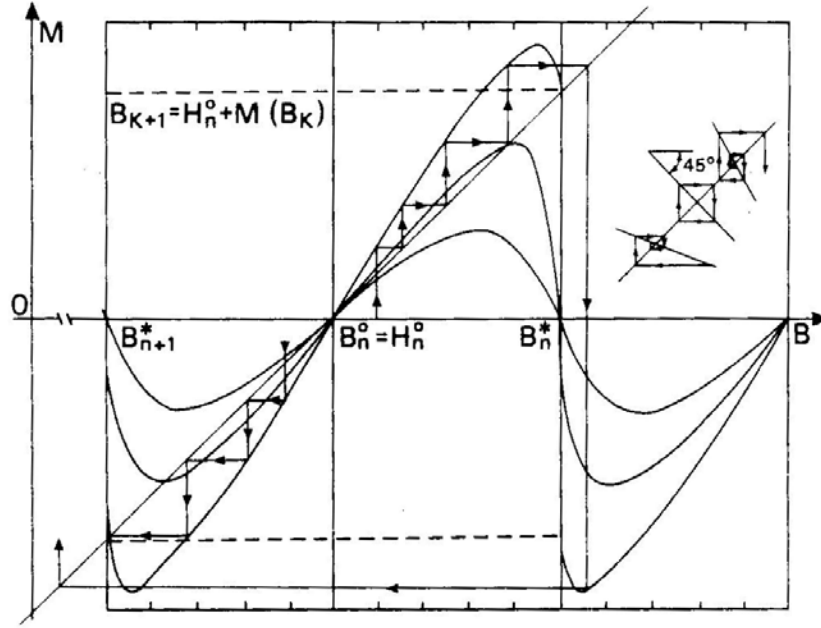
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Figure 20: Schematic construction of the logistic map in ideally conducting phases.

Now the wave equation takes the form

$$\sigma E = \frac{ic^2}{4\pi\omega}\left[k \times (k \times E)\right] - 4\pi\left[k \times \chi\left(k \times E\right)\right] \tag{60}$$

and has an additional to equation term

$$j_m = \frac{ic^2}{\omega}4\pi\left[k \times \chi\left(k \times E\right)\right] \tag{61}$$

and we arrive at

$$\left[\sigma_{xx} + \left(\frac{ic^2k^2}{4\pi\omega}\right)\left(1 - 4\pi\frac{\partial M}{\partial B}\sin^2\theta\right)\right]E_x + \sigma_{xy}E_y + \sigma_{xz}E_z = 0 \tag{62}$$

The difference can be accounted by substitution

$$\sigma_{ik} \rightarrow \widetilde{\sigma}_{ik} = \sigma_{ik}\left(1 - 4\pi\frac{\partial M}{\partial B}\sin^2\theta\right)^{-1}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z = 0 \tag{63}$$
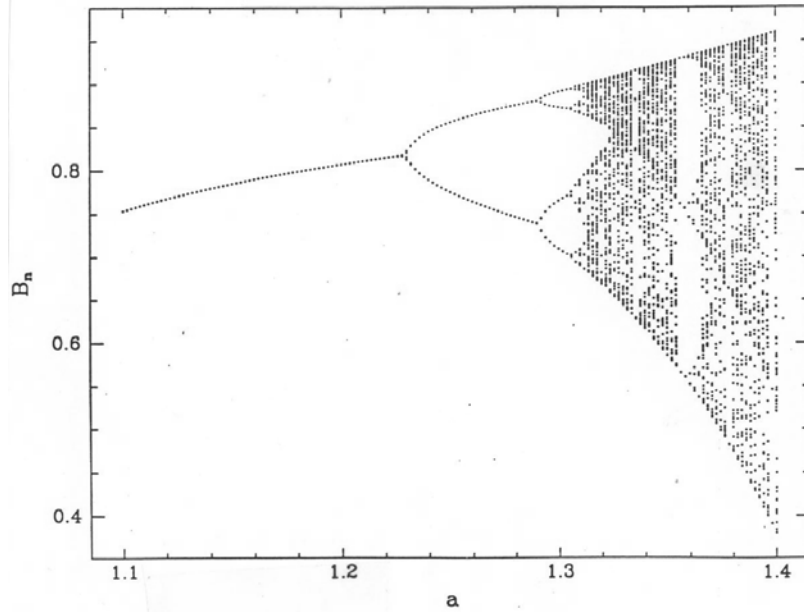
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Figure 21: Bifurcation patern in ideally conducting phases.

5.2 Relativistic invariance of ICP and of quantum Hall effect

Relativistic and scaling invariance of a non-dissipative two-dimensional electron system under strong magnetic fields, which is the case in ICP and QHE systems can be obtained from the following simple phenomenological consideration [7].

Under very general conditions, in the theory of continuous media the induced charges and currents are linearly connected with electric and magnetic fields:

$$J_\mu = \Xi_{\mu\nu\rho} F^{\nu\rho} \tag{64}$$

where j_μ is the current, $F_{\nu\rho}$ is the electromagnetic field strength tensor, and $\Xi_{\mu\nu\rho}$ are the coefficients, defining the physical properties of a conducting media. The coefficients express the linear response of the medium (conductor, insulator, ferromagnet, etc.) to applied fields. In all the usual condensed matter cases medium destroys relativistic invariance of Maxwell equations. In the vacuum, which is a relativistically invariant medium, $\Xi_{\mu\nu\rho}$ is zero. If the coefficients $\Xi_{\mu\nu\rho}$ form an invariant tensor: $\Xi'_{\mu\nu\rho} = \Xi_{\mu\nu\rho}$ where prime corresponds to a moving frame, the medium would preserve the relativistic invariance. In D + 1 space-time dimensions the only constant tensors are

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the metric tensor $g_{\mu\nu}$ and the totally antisymmetric D + 1 component tensor $\epsilon_{\mu\nu}$ ... . In 3 + 1 dimensions it is impossible to construct a nonvanishing constant three component tensor, out of $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. Therefore the only relativistically invariant medium in 3+1 dimensions is the vacuum, where $\Xi_{\mu\nu\rho\sigma} = 0$.

However in 2 + 1 dimensions such a tensor may exist: $\sigma_{\mu\nu\rho} \propto \epsilon_{\mu\nu\rho}$. Let us show that this situation is realized under the QHE conditions.

Indeed, in the QHE medium (in the plateau regime) the material equations are:

$$j_1 = \sigma_{xy}E_y \equiv \nu\alpha E_2; \qquad j_2 = -\sigma_{xy}E_x \equiv -\nu\alpha E_1 \qquad (65)$$

where ν is the filling factor and α is the fine structure constant. These two equations for the Hall currents could be cast in a form:

$$j_i = \nu\epsilon_{ij0}E^j \qquad (66)$$

where $i, j = 1, 2$. The zero component of current

$$j_0 = \nu\alpha B_3 \qquad (67)$$

is, in fact, the Faraday law for a nondissipative medium: $\int j_0 \propto Q \equiv \nu\alpha \int B_z$.

Eqs.(66,67), at $\nu = 1$, could be cast in a form:

$$j^i = 2c\alpha e^{ijk} F_{jk} \qquad (68)$$

Eq. (68) follows also from an action:

$$S_A \propto \int d^3x \left[-\frac{1}{c}j^i A_i + \alpha e^{ijk} A_i F_{jk} \right].$$

Indeed, the variation of this action:

$$\delta S_A \propto \left[-\frac{1}{c}j^i A_i + 2\alpha e^{ijk} A_i F_{jk} \right] \delta A_i = 0 \qquad (69)$$

yields Eq. (68).

The mentioned above Lorentz invariance of QHE is connected with the Chern-Simons term, which is a new, compared to 3D electrodynamics, term in Lagrangian.

Inserting Eqs. (66,67) into the effective Maxwell equations, we obtain the macroscopic equations in medium in Fourier space [6, 7]:

$$k_\mu F_{\mu\nu} = \nu\alpha\epsilon_{\mu\nu\rho}\sqrt{k^2}F_{\nu\rho}; \qquad \epsilon_{\mu\nu\rho}\partial_\mu F_\nu\rho = 0 \qquad (70)$$

or

$$\partial^{\mu} F_{\mu\nu} = \nu\alpha\epsilon_{\mu\nu\rho}\sqrt{\partial^2} F^{\nu\rho} \tag{71}$$

The Lagrangian, generating these equations is:

$$\mathcal{L} = \frac{1}{4} f_{\alpha\beta} \frac{\theta(\partial_\alpha \partial^\alpha)}{\sqrt{\partial_\alpha \partial^\alpha}} f^{\alpha\beta} + \nu\epsilon_{\alpha\beta\gamma} a^\alpha f^{\beta\gamma} \tag{72}$$

where the first term is the vacuum Lagrangian, and the second is the Chern-Simons term.

It can be easily shown explicitly by performing Lorentz transformation for the currents and fields, that the nondiagonal components of the conductivity tensor preserves the Lorentz invariance while the diagonal components destroy it. This will be elaorated in details in what follows.

5.3 Composite Fermions-Chern-Simons gauge transformed particles

It is known that the integer quantum Hall effect ($IQHE$) can be qualitatively understood on the basis of the single electron wave functions in the electrostatic potential of the long range fluctuations in th sample, the fractional quantum Hall effect ($FQHE$) is essentially a many-body phenomenon, and was extensively studied, mainly in the framework of the "Laughlin wave function" formalism [60]. A very popular model of *Composite Fermions* was suggested by Jain, who used a very transparent picture of magnetic fluxes attached to an electron, in the spirit of the Faraday magnetic flux lines piercing the conducting media.

The vector potential, associated with this flux tube (solenoid) may be written as:

$$\vec{a}(\vec{r}) = \frac{\Phi}{2\pi} \frac{\vec{z} \times \vec{r}}{r^2} \tag{73}$$

where $\vec{z}$ is a unit vector along the z - axis. The magnetic field B, associated with such vector potential is that of a vortex of strength Φ, localized at the origin:

$$\vec{B} = curl\vec{a} = \Phi\delta(\vec{r}) \tag{74}$$

where the δ-function arises from the singularity, at $\vec{r} = 0$ of the vector potential, Eq. (73). The magnetic flux, connected to such field is:

$$\int \vec{B} d\vec{s} = \Phi \tag{75}$$

The Schrödinger equation for a particle in such a field, is:

$$\hat{H}(\Phi)\psi(r) = \frac{1}{2m}\left(\vec{\mathcal{P}} - \frac{e}{c}\vec{a}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}) \tag{76}$$

where the canonical momentum operator $\vec{\mathcal{P}}$ is:

$$\vec{\mathcal{P}} = -i\hbar\frac{\partial}{\partial\vec{r}} \tag{77}$$

The wave function $\psi(\vec{r})$ should be periodic under rotations on the angle $\theta = 2\pi$ around z-axis.

For the flux-tube-particle composite, the spectrum of the angular momentum is:

$$\ell_z = \hbar\left(m - \frac{\Phi}{\Phi_0}\right) \tag{78}$$

Following Wilczek, the angular momentum of this composite particle is equal to

$$s = \frac{\ell_z(m=0)}{\hbar} = -\frac{\Phi}{\Phi_0} \tag{79}$$

In general, s is neither integer nor half integer, and can take any value. If we interchange flux-tube-particle composites we will have an additional phase factor. Since the interchange of two such composites can give *any* phase, Wilczek called them *anions*.

For a system of N flux-tube-particle composites, the charged particle feels the vector potential of flux tubes, "glued" to all the other particles. The corresponding Hamiltonian can be written as follows:

$$\hat{H} = \frac{1}{2m}\sum_{i=1}^{N}\left(\vec{\mathcal{P}} - \frac{e}{c}\vec{a}_i(\vec{r}_i, \ldots \vec{r}_N)\right)^2 \tag{80}$$

with the vector potential:

$$\vec{a}_i = \frac{\Phi}{2\pi}\sum_{j\neq i}\frac{\vec{z}\times(\vec{r}_i - \vec{r}_j)}{(\vec{r}_i - \vec{r}_j)^2} = \frac{\Phi}{2\pi}\sum_{j\neq i}\nabla\theta_{ij} \tag{81}$$

where θ_{ij} is the angle between the vector $\vec{r}_i - \vec{r}_j$ and the x-axis. This effective vector potential is nonlocal, since it depends on the position of all particles and, in particular, it vanished at $N = 1$. The Hamiltonian, Eq. (80), describes the interaction of charged particles with the Chern-Simons
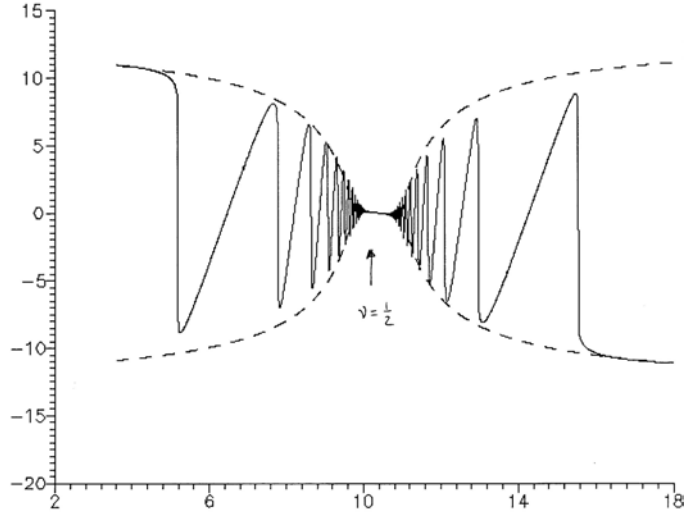
Figure 22: Magnetization oscillations in composite fermions. Note the symmetry aound the $\nu = \frac{1}{2}$ filling factor.

gauge field, and this vector-potential corresponds to the following effective magnetic field:

$$\vec{B}'_i = \vec{B} - \Phi \sum_{j \neq i} \delta(\vec{r}_i - \vec{r}_j) \tag{82}$$

which can be presented as

$$B_{eff} = B - \frac{\Phi}{\Phi_0} \frac{2\pi\hbar c}{e} n_e = B\left(1 - \frac{\Phi}{\Phi_0} 2\pi\ell_B^2 n_e\right) = B\left(1 - \frac{\Phi}{\Phi_0}\nu\right) \tag{83}$$

As it follows from Eq. (83), the effective magnetic field B_{eff} is equal to zero for the magnetic flux $\Phi = \frac{\Phi_0}{\nu}$. The case of the filling factor $\nu = \frac{1}{2n}$ is of special interest, because for $\Phi = 2n\Phi_0$ the statistics of particles is not changed. A very interesting situation corresponds to the filling factor $\nu = \frac{1}{2}$, when the dimensionless magnetic flux

$$\tilde{\Phi} = \frac{\Phi}{\Phi_0} \tag{84}$$

is equal to 2. The completely filled Landau level with $m = 1$ and $B_{eff} = 0$ corresponds to $\tilde{\Phi} = 1$.
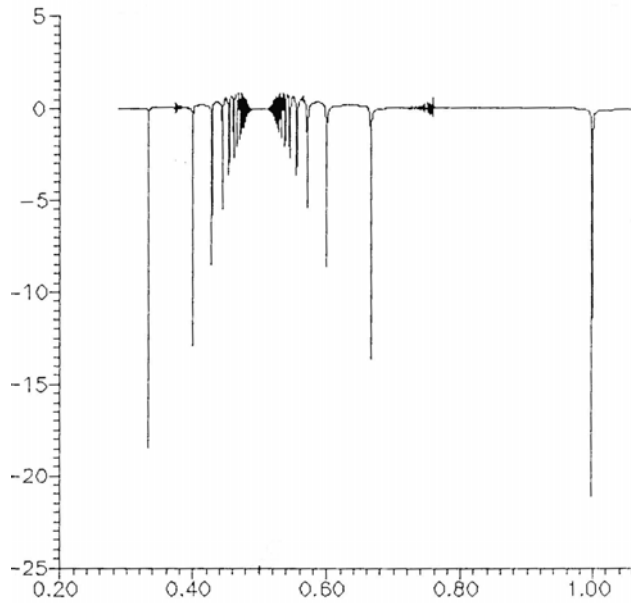
Figure 23: Susceptibility oscilations in composite fermions.

6 Lattice effects in correlated systems

The field of highly correlated systems such as heavy fermions and high -T_c copper oxides have been attracting considerable attention during the last two decades. The presence of strong on-site Coulomb repulsions distinguishes these systems from other large family of traditional superconductors where phonons play a dominant role. This has motivated experimental activities in measuring the de Haas-van Alphen (dHvA) effect in strongly correlated systems [64-67].

There are two main difficulties in observing the dHvA effect in high -T_c superconductors. First, within the accessible range of stationary magnetic fields, the Larmor radii in the copper-oxygen plane is roughly of the order of magnitude of the mean free path of the existing samples. Second, a magnetic field exceeding the upper critical field H_{c2} is needed to reach the normal state. For example, H_{c2} for $YBa_2Cu_3O_7$ is in the range [67] of 100T in low temperatures. The improvements in the sample quality and the development of high field magnets may overcome the difficulties in future.

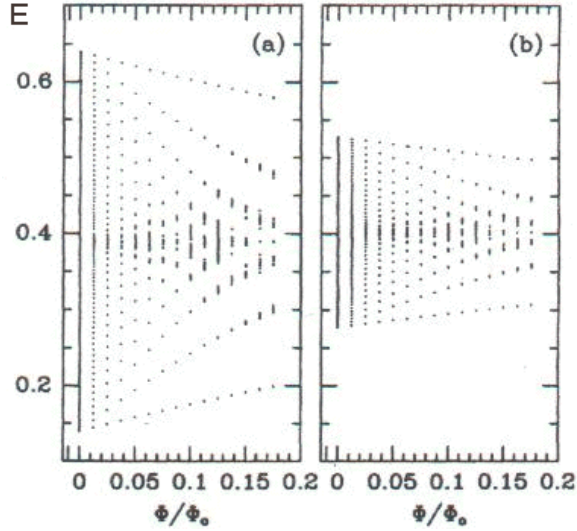Despite these difficulties, several attempts have been made to observe

Figure 24: The Hofstadter fractal spectrum (the butterfly) for the Landau electron in a periodic potential.

the dHvA effect in the $YBa_2Cu_3O_7$ compound [67]. These experiments show some features of the Fermi surface predicted from band structure calculations based on the local-density-approximation (LDA). A study of the dHvA effect based on the marginal Fermi liquid theory suggests [21] that when the frequency dependent effective mass is introduced the reduction in amplitude by self-energy term is similar to that of canonical Fermi liquid theory. This work, however, does not account for the contributions from the lattice and Coulomb correlations.

Any analysis of the dHvA effect data by using the standard Liftshitz-Kosevich form [68] which neglects the lattice and the Coulomb correlation effects may not be complete because the copper-oxide system is described by a tight-binding model, and strong on-site Coulomb correlations are considered relevant interactions for the system.

The lattice effects on the dHvA oscillations in the presence of strong Coulomb correlations have been considered in [22]. The lattice effects are manifested in both the periodic potential, and the coupling between a quasiparticle and the lattice distortion. The former is described by the tight-binding band structure, and the latter is described by the electron-phonon interaction. In what follows, the terms *potential* and *interaction effects* refer to the contributions from the periodic potential and the electron-phonon
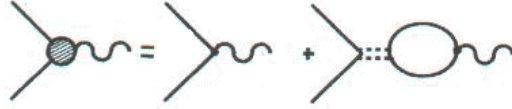
Figure 25: The Feynman diagrams for the Coulomb screened electron-phonon coupling constant. The first and the second diagram on the right-hand side refer to the contributions from the renormalized band structures and the dynamical screening effects, respectively.

interaction, respectively. It is assumed in [22] that the contribution of Coulomb correlations is to renormalize the band structure and to screen the electron-phonon interaction. Hence, the term *Coulomb correlation effects* is used to refer to the Coulomb renormalized band structure effects.

Due to the underlying lattice, the motion of electron in the magnetic field is restricted, and, as a consequence, the states in a tight-binding band (in zero field) split [69] into a series of magnetic subbands called *Hofstadter spectrum*. Because the number of states is equally partitioned in each magnetic subbands, the field dependence of these subbands are complex due to the presence of a van Hove singularity in the density of states

$$N(E) \; = \; \frac{1}{(2\pi)^2} \int_E \frac{d\mathbf{k}}{v_{\mathbf{k}}} \tag{85}$$

at half filling. (Half filling, in the Fermi liquid picture, corresponds to zero hole doping concentration, x.) Here, $v_{\mathbf{k}}$ is the quasiparticle velocity. It is noted that Coulomb correlations renormalize the tight-binding band via shifting the center of the band upward in energy by λ_o and via reducing the bandwidth from $4t$ to $4t^*$. Hence, near half filling, both the *potential* and the *Coulomb correlation effects* significantly change the dHvA effect from the free electron-like behavior. The deviation from the free electron-like behavior in **low** fields is measured by the filling fraction, $\nu = \frac{1}{2}(1 - \mathrm{x})$, dependence of the dHvA effect and by the magnetic field dependence of the effective mass. (Here, the term **low** refers to the low field regime of the Hofstadter spectrum where the magnetic subbands are approximated as single energy levels. This regime corresponds to the field strength much less than one flux quantum through a plaquette.)

Both the *potential* and the *Coulomb correlation effects* lead to the mass enhancement as a function of ν. The *potential effect* leads to the increase in the effective mass as the van Hove singularity is approached, while the

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Coulomb correlation effects lead to the diverging mass via Brinkman-Rice localization [70] as half filling is approached.

In [22], for simplicity, the one-band Hubbard Hamiltonian is applied to a square lattice as a model for correlated systems. The Coulomb correlations in the infinite U limit are considered by using the slave boson technique and the systematic $1/N$ expansion scheme is employed in determining the Coulomb screened electron-phonon coupling constant self-consistently in order to estimate the *interaction effect*.

To employ the auxiliary bosonization scheme [72], the infinite Coulomb correlation U limit can be chosen and the Hubbard Hamiltonian can be written as:

$$H = - \sum_{<ij>,\sigma} t_{i,j} (C_{i,\sigma}^{\dagger} \mathbf{e}_i \mathbf{e}_j^{\dagger} C_{j,\sigma} + h.c.) \tag{86}$$

where the operators $C_{i,\sigma}^{\dagger}$ and $\mathbf{e}_i^{\dagger}$ create the half-full and empty occupation state at the site i, respectively. The matrix element $t_{i,j}$ denotes the effective hopping between nearest neighboring copper sites through an oxygen ion. In the $U = \infty$ limit, only the half-full and empty occupation states are allowed; therefore, the constraint

$$\sum_{\sigma} C_{i,\sigma}^{\dagger} C_{i,\sigma} + \mathbf{e}_i^{\dagger} \mathbf{e}_i = 1 \tag{87}$$

is imposed via a Lagrange multiplier λ_i to forbid double occupancy at each sites. In this way, the Coulomb correlation effects are described by $\mathbf{e}_i$ and λ_i.

In magnetic fields, Bose operators $\mathbf{e}_i^{\dagger}$ and $\mathbf{e}_i$, and the Lagrange multiplier remain unchanged, but the hopping matrix element acquires a phase term

$$t_{i,j} \rightarrow t_{i,j} e^{i\frac{2\pi}{\phi_o} \int_i^j \mathbf{A} \cdot \mathbf{dl}} \tag{88}$$

where $\mathbf{A}$ is a vector potential. For convenience, the Landau gauge $\mathbf{A} = B(0, x, 0)$ is chosen with the externally applied uniform magnetic field pointing in the z-direction. In mean field theory, the energy spectrum is computed by replacing Bose operators and Lagrange multiplier with their saddle point values ($< \mathbf{e}_i^{\dagger} >=< \mathbf{e}_i >= e_o$ and $\lambda_i = \lambda_o$)

$$H = -e_o^2 t \sum_{<ij>\sigma} (e^{i\frac{2\pi}{\phi_o} \int_j^i \mathbf{A} \cdot dl} C_{i,\sigma}^{\dagger} C_{j,\sigma} + h.c.) + \lambda_o \sum_i (\sum_{\sigma} C_{i,\sigma}^{\dagger} C_{i,\sigma} + e_o^2 - 1) \ . \tag{89}$$

The variational parameters e_o and λ_o indicate that Coulomb correlations lead to band renormalization, and their effects become strong as x ($\propto e_o^2$) is decreased.

In Landau gauge, Hamiltonian Eq. (89) is invariant under the translation $(y \rightarrow y + a)$ in the y- direction; however, it is not invariant with the translation $(x \rightarrow x + a)$ in the x- direction. Here, a is the lattice parameter. This is the consequence of our choice of gauge in which a magnetic field affects, only, the motion of electrons in the x- direction. When the flux ϕ, in units of ϕ_o, through a plaquette of area a^2 is a rational fraction p/q (both p and q are prime integers), the Hamiltonian is invariant with the translation $(x \rightarrow x + qa)$. This translation corresponds to a magnetic superlattice with a primitive cell $(q \times 1)a$. In the Bloch diagonalization of Eq. (89) in both the x and y-directions, the Fourier representation of the half-full occupation operator

$$C_{\mathbf{i}} \;=\; \frac{1}{\sqrt{N'_x N_y}} \sum_{\mathbf{k}} C_{\mathbf{k}}^{\gamma} e^{i\mathbf{k}\cdot\mathbf{i}a} \tag{90}$$

is substituted into the Hamiltonian. Here, $N'_x = N_x/q$, and the magnetic Brillouin zone is given by $(-\pi/qa \leq k_x \leq \pi/qa, -\pi/a \leq k_y \leq \pi/a)$. When the sum is taken over the site index $\mathbf{i} = (i_x, i_y)$ in Eq. (89), the Hamiltonian becomes

$$H \;=\; N\{\sum_{\mathbf{k}}\sum_{\gamma=1}^{q} C_{\mathbf{k}}^{\gamma\dagger} C_{\mathbf{k}}^{\gamma}[\lambda_o - 2\frac{r_o^2}{t}\cos(k_y a - 2\pi\frac{\phi}{\phi_o}\gamma)] \;-$$

$$-\frac{r_o^2}{t}\sum_{\mathbf{k}}\sum_{\gamma=1}^{q}[C_{\mathbf{k}}^{\gamma\dagger} C_{\mathbf{k}}^{\gamma+1} e^{ik_x a} + C_{\mathbf{k}}^{\gamma\dagger} C_{\mathbf{k}}^{\gamma-1} e^{-ik_x a}] \;+\; \lambda_o(\frac{r_o^2}{t^2} - \frac{1}{N})\} \tag{91}$$

where $r_o = e_o t/\sqrt{N}$, and the index γ denotes the position within a magnetic supercell. Here, the sum over the spin index σ leads to the degeneracy factor N. The energy spectrum obtained by diagonalizing the Hamiltonian Eq. (91) is the Hofstadter spectrum for strongly correlated electrons.

Due to the small lattice constant, extremely strong magnetic fields (for a =4A, $\phi/\phi_o = 1$ corresponds to B=2.6$\times 10^4$T) are needed to observe a complete Hofstadter spectrum in the cuprates, but for the purpose of the dHvA effect calculation, the **low** field behavior of magnetic subbands is most relevant. Following two properties of magnetic subbands are worth discussing at this moment. First, the electronic states (in the zero field) are quenched to form magnetic subbands, and there is gap between two adjacent subbands. Due to the Van Hove singularity, the size of gap is not uniform and has nonlinear dependence on both the filling fraction and the field strength. The energy gap shrinks monotonically as either the filling fraction $\nu = \frac{1}{2}$ is approached or the magnetic field is decreased. Second, the magnetic

subbands away from the band center appear as single degenerate energy levels. In low fields, there are many subbands that behave as single levels; hence the oscillation in magnetization becomes sharp. As the magnetic flux through a plaquette decreases (i.e., $q \to \infty$), the number of magnetic subbands that behave as single levels increases since the overall number of magnetic subbands increases, and their bandwidth narrows. (As $q \to \infty$, the ratio of energy gap to the bandwidth vanishes [69].) In **low** fields, lowest several magnetic subbands are virtually identical to Landau levels. For this reason, when the filling fraction is low, the *potential effect* is absent, but it becomes pronounced as half filling is approached.

The nonlinear field dependence of the energy gap due to the underlying lattice leads to the ν dependence of the effective mass. This is easily demonstrated in **low** fields. In the long magnetic length l_m limit ($l_m = a\sqrt{q/2} >> a$ corresponding to $\phi/\phi_o << 1$), a simple expression for energy levels is obtained by solving the Schrödinger equation

$$H \mid \Psi > \; = \; E \mid \Psi > \; . \tag{92}$$

When the Hamiltonian Eq. (91) acts on the eigenstate

$$\mid \Psi > \; = \; \sum_{\gamma=1}^{q} g_{\mathbf{k}}^{\gamma} C_{\mathbf{k}}^{\gamma\dagger} \mid 0 >,$$

Eq. (92) becomes the difference equation

$$[\lambda_o - 2\frac{r_o^2}{t}\cos(k_y a - 2\pi\frac{\phi}{\phi_o}\gamma)]g_{\mathbf{k}}^{\gamma} - \frac{r_o^2}{t}[g_{\mathbf{k}}^{\gamma+1}e^{ik_x a} + g_{\mathbf{k}}^{\gamma-1}e^{-ik_x a}] = g_{\mathbf{k}}^{\gamma}E \; . \tag{93}$$

In the continuum limit, Eq. (93) is written as the equation of motion for a harmonic oscillator, [71] and eigenenergy is given by

$$E_n \; = \; \Xi \; + \; \hbar\omega_c(n + \frac{1}{2}) \tag{94}$$

where

$$\Xi \; = \; [\lambda_o - t(2 \; + \; \frac{\cos^2 k_F a + 1}{\cos k_F a})]\frac{Nr_o^2}{t^2} \; + \; \lambda_o(N-1) \; . \tag{95}$$

Here, $\omega_c = eB/m^*c$ is the cyclotron frequency, and k_F denotes the Fermi vector in the y-direction. The eigenenergy Eq. (94) for the **low** field Hofstadter spectrum differs from a series of Landau levels in the following ways: the energy levels are shifted by Ξ, and the separation between neighboring

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energy levels depends on the x dependent effective mass m^*. The x dependence of m^* which arises from both the *potential* and *Coulomb correlation effects* is reflected in the expression of the renormalized cyclotron mass

$$m^* = \frac{\hbar^2 t}{2a^2 N r_o^2} \cdot \frac{1}{(\cos k_F a)^{1/2}} \tag{96}$$

where the factors $(\cos k_F a)^{-1/2}$ and r_o^{-2} result from the *potential* and *Coulomb correlation effects*, respectively. Due to the *potential effect*, the divergence of m^* in Eq. (96) as x$\to$ 0 is stronger than from the *Coulomb correlation effect* alone.

6.1   Magnetization oscillations

The resemblance between the magnetic subbands (in **low** fields) and a series of Landau levels can be used to calculate the magnetization curve. As we have discussed previously, there are two notable differences between the lattice and the free electron gas systems. Namely, the energy gap between two adjacent levels is not uniform, and the field dependence of magnetic subbands is not linear. These differences lead to the magnetic field and the x dependence of cyclotron mass, chemical potential, and magnetization.

In the Luttinger-Bychkov-Gorkov formalism [73] the thermodynamic potential is

$$\Omega = -\frac{1}{\beta} \sum_{i\omega_n} Tr\{ln[-G^{-1}(k)] + \Sigma(k)G(k)\} + \Omega^{'}\{G\} \tag{97}$$

where $\beta = 1/k_B T$, $G(k) = 1/(i\omega_n - E_{\mathbf{k}} - \Sigma(k))$ is the proper quasiparticle Green function, and $\Sigma(k)$ is the electronic contribution to self-energy. Here, k denotes the three momenta $(\mathbf{k}, i\omega_n)$ where $i\omega_n = i(2n+1)\pi/\beta$ is the discrete Matsubara frequency for fermions. It has been suggested [15] that, in strongly interacting electron systems in three dimensions, this self-energy contribution in Eq. (97) leads to the deviation from the Liftshitz-Kosevich form for the magnetization curve.

Because the electron self-energy term in the Fermi liquid approximation does not [74] remove the energy gap in two dimensions, this term which arise as the order O(1/N) contribution in the perturbation theory is not considered in the calculation of thermodynamic potential. However, some aspects of Coulomb correlation effect is contained in the calculation. For instance, the correlation effects are included in the mean field definition of

quasiparticles. In mean-field theory, the thermodynamic potential in Eq. (97) simplifies to

$$\Omega = -\frac{1}{\beta} \sum_{i\omega_n} Tr \ln[-G_o^{-1}(k)] \tag{98}$$

where the bare Green function is written as $G_o(k) = 1/(i\omega_n - E_{\mathbf{k}})$, and the Coulomb renormalized band structure is given by

$$E_{\mathbf{k}} = \lambda_o - \mu - \frac{2r_o^2}{t}(\cos k_x a + \cos k_y a) . \tag{99}$$

Here, μ denotes the chemical potential. In magnetic fields, the sum over the band index n and the momentum $\mathbf{k}$ (within the magnetic Brilliouin zone) replaces the Tr in Eq. (98), and the thermodynamic potential is given by

$$\Omega = \frac{2}{\beta} \sum_{\mathbf{k}} \sum_{n=0}^{n_D-1} ln[1 + e^{-\beta(E_n(\mathbf{k})-\mu)}] . \tag{100}$$

Here, $n_D = q$ denotes the number of magnetic subbands in the field ϕ/ϕ_o ($= p/q$), and the factor 2 accounts for the spin degeneracy. Because the magnetic subbands that are away from the center of the spectrum have narrow bandwidth [69], the Hofstadter spectrum can be approximated as a series of energy levels and replace the $\mathbf{k}$-space sum by

$$\sum_{\mathbf{k}} \rightarrow \int dE_{\mathbf{k}} \delta(E_n - E_{\mathbf{k}}) , \tag{101}$$

where E_n is the eigenenergy (94). Within this approximation, the *skeletons* and the limited fine structure of the subcells for the Hofstadter spectrum [69] could be calculated in the following way. The thermodynamic potential, in this approximation, reduces to

$$\Omega = \frac{2g(B)}{\beta} \sum_{n=0}^{n_D-1} ln[1 + e^{-\beta(E_n-\mu)}], \tag{102}$$

where $g(B) = N_x N_y/q$ denotes the degeneracy of a band.

The magnetization (per unit area) M curve for a fixed number of particles is easily computed from the free energy, $F = \Omega - \mu N$, and it is given by

$$M = 2\tilde{g}(B) \sum_{n=0}^{n_D-1} f(E_n)\frac{\partial E_n}{\partial B} , \tag{103}$$

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where $\tilde{g}(B) = 1/n_D$, and $f(E)$ is the Fermi function. The chemical potential is redefined: $\tilde{\mu} = \mu - \Xi$ and the eigenenergy is:

$$\tilde{E}_n = \frac{\hbar e B}{m_n^*(B)c}(n + \frac{1}{2}) \quad . \tag{104}$$

In Eq. (104), the field dependent effective mass $m_n^*(B)$ is introduced in order to describe the deviation of magnetic subbands from the Landau level-like behavior. It is noted that, in free electron-like systems, the effective mass does not depend on either the band index or the magnetic field. Because m_n^* measures the band curvature, and the Hofstadter spectrum has a reflection symmetry about the band center, $m_n^* = -m_{(n_D-1)-n}^*$. Hence, only the filling fraction $\nu < \frac{1}{2}$ should be considered in the calculation of effective mass.

In two dimensions, the chemical potential oscillates strongly with the field [8] that this must be computed and must be incorporated in the calculation of thermodynamic properties. For a fixed value of x, the chemical potential can be computed from

$$\frac{n_D}{2}(1 - \text{x}) = \sum_{n=0}^{n_D-1} [e^{\beta(\tilde{E}_n - \tilde{\mu})} + 1]^{-1} \quad . \tag{105}$$

6.2   External perturbations in quasi-two dimensional model

Here the influence of a general external perturbation on the magnetization oscillation in quasi-two dimensions is estimated. In the present discussion, we refer any strongly anisotropic systems where the hopping matrix element t_z in the z-direction (parallel to the field) is small compared to the overlap integral t in the xy-plane (perpendicular to the field) as quasi-two dimensional systems and write the dispersion relation in **low** magnetic fields as

$$E_n(k_z) = -2t_z \cos k_z d + \Xi + \hbar \omega_c (n + \frac{1}{2}) , \tag{106}$$

where d is the lattice parameter in the z-direction. In this case the variational parameters e_o and λ_o in Eq. (106) are approximated by replacing them with the saddle point values in two dimensions. It can be assumed: $t_z >> \hbar \omega_c$, so that there is no energy gap in the spectrum, and the large number of magnetic subbands crosses the Fermi energy. In contrast to the case in two dimensions, the oscillation of chemical potential, in quasi-two dimensions, is weak that it can be approximated as independent of the field.

When the quasiparticles are coupled to an external perturbation, the self-energy Σ_{ex} contribution from the interaction dresses the bare Green

function. Hence, the thermodynamic potential Eq. (98) is written as

$$\Omega = -\frac{1}{\beta}\sum_{i\omega_n} Tr \ ln[-i\omega_n + E_{\mathbf{k}} + \Sigma_{ex}(k)] \ . \qquad (107)$$

In summing the Matsubara frequency $i\omega_n$ in Eq. (107), it is convenient to separate the self-energy contribution into the real and imaginary components by analytically continuing onto the real frequency axis

$$\Sigma_{ex}(\omega \pm i\delta) = \Sigma_R(\omega) \mp i\Sigma_I(\omega) \ . \qquad (108)$$

Substituting the self-energy Eq. (108) into the thermodynamic potential Eq. (107) and rewriting Ω in magnetic fields yields

$$\Omega = 2g\sum_{k_z,n}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}f(\omega)\ tan^{-1}\frac{\Sigma_I(\omega)}{\omega - E_n(k_z) - \Sigma_R(\omega)} \qquad (109)$$

where n denotes the magnetic subband index. In Eq. (109), it is noted that the approximation (101) is used to replace the momentum sum with an energy integral. By exploiting the similarity between a series of Landau levels and magnetic subbands in **low** fields (i.e., $q \to \infty$), one can follow the work [75] of Engelsberg and Simpson. By using the Poisson sum formula

$$\sum_{n=0}^{\infty}F(n) = Re\{\ \int_0^{\infty}dxF(x)[1 + 2\sum_{r=1}^{\infty}(-1)^r e^{i2\pi rx}]\ \} \ , \qquad (110)$$

we approximate the discrete sum over n in Eq. (109) with a continuous integral and the sum over Fourier components r. The substitution of Eq. (110) into Eq. (109) and the change of variable $E' = \hbar\omega_c x$ yields the following expression for the thermodynamic potential

$$\begin{aligned}\Omega &= 2gRe\{\sum_{r=1}^{\infty}(-1)^r\int_{-\pi/d}^{\pi/d}\frac{dk_z}{2\pi}\int\frac{d\omega}{\pi}f(\omega)\\ &\int_0^{\infty}\frac{dE'}{\hbar\omega_c}e^{i\frac{2\pi rE'}{\hbar\omega_c}}tan^{-1}\frac{\Sigma_I}{\omega - E_z - E' - \Sigma_R}\ \}\end{aligned} \qquad (111)$$

where $E_z = -2t_z cosk_z d - \widetilde{\mu}$. The E' integration in Eq. (111), is transformed into a simple contour integral by following three steps: first, apply the integration by parts method, second, ignore the small end-point contributions, and finally, extend the lower limit of the integration from $-\widetilde{\mu}$ to $-\infty$. It is noted that the last step leads to a small change in the thermodynamic

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potential, but we ignore this difference since it does not affect the oscillatory part of the potential. Performing the integration over the contour and over k_z in Eq. (111), we simplify the potential to

$$\Omega = -2g\hbar\omega_c Re\{\sum_{r=1}^{\infty}\frac{(-1)^r}{r}e^{i\frac{2\pi r\tilde{\mu}}{\hbar\omega_c}}J_o(\frac{2\pi rt_z}{\hbar\omega_c})\int\frac{d\omega}{2\pi i}f(\omega)e^{i\frac{2\pi r}{\hbar\omega_c}(\omega-\Sigma_R+\Sigma_I)}\ \}$$

(112)

where the Bessel function J_o arises from the k_z integration. The magnetization is computed from Eq. (112) by fixing the chemical potential

$$M = \frac{e\tilde{\mu}}{\pi\hbar c\beta}\sum_{r=1}^{\infty}(-1)^r cos(\frac{2\pi r\tilde{\mu}}{\hbar\omega_c})J_o(\frac{2\pi rt_z}{\hbar\omega_c})\sum_{l=0}^{\infty}e^{-\frac{2\pi r}{\hbar\omega_c}[\omega_l+\zeta(\omega_l)]}$$

(113)

where $\zeta(\omega_n) = i\Sigma_{ex}(i\omega_n)$. The dominant contribution to the magnetization curve Eq. (113) comes from the first Fourier harmonic ($r = 1$). In Eq. (113), the influence of an external perturbation as described by the self-energy term modifies the oscillation amplitude. This effect is identical to the enhancement of cyclotron mass via the self-energy term. This suggests that when external perturbations are coupled to a quasiparticle, their effects reduce the amplitude; therefore, as the coupling increases, the amplitude decreases.

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